timber products include three categories: pulpwood, plywood, and sawlogs. Several reforestation alternatives are available for each site, each with its cost, number of rotation years (i.e., number of years from seedling size until harvesting), return from rent, and production output. The following table summarizes this information.

		Annual \$/acre		Rotation	Annual m^3/acre		
Site	*Alternative*	*Cost*	*Rent*	*yr*	*Pulpwood*	*Plywood*	*Sawlogs*
1	*A*1	1000	160	20	12	0	0
	*A*2	800	117	25	10	0	0
	*A*3	1500	140	40	5	6	0
	*A*4	1200	195	15	4	7	0
	*A*5	1300	182	40	3	0	7
	*A*6	1200	180	40	2	0	6
	*A*7	1500	135	50	3	0	5
2	*A*1	1000	102	20	9	0	0
	*A*2	800	55	25	8	0	0
	*A*3	1500	95	40	2	5	0
	*A*4	1200	120	15	3	4	0
	*A*5	1300	100	40	2	0	5
	*A*6	1200	90	40	2	0	4
3	*A*1	1000	60	20	7	0	0
	*A*2	800	48	25	6	4	0
	*A*3	1500	60	40	2	0	4
	*A*4	1200	65	15	2	0	3
	*A*5	1300	35	40	1	0	5

To guarantee sustained future production, each acre of reforestation in each alternative requires that as many acres as years in rotation be assigned to that alternative. The rent column represents the stumpage value per acre.

The goals of Warehouzer are as follows:

1. Annual outputs of pulpwood, plywood, and sawlogs are 200,000, 150,000, and 350,000 cubic meters, respectively.
2. Annual reforestation budget is \$2.5 million.
3. Annual return from land rent is \$100 per acre.

How much land at each site should be assigned to each alternative?

8.2 A charity organization runs a children's shelter. The organization relies on volunteer service from 8:00 A.M. until 2:00 P.M. Volunteers may begin work at the start of any hour between 8:00 A.M. and 11:00 A.M. A volunteer works a maximum of 6 hours and a minimum of 2 hours, and no volunteers work during lunch hour between 12:00 noon and 1:00 P.M. The charity has estimated its goal of needed volunteers throughout the day (from 8:00 A.M. to 2:00 P.M., and excluding the lunch hour between 12:00 noon and 1:00 P.M.) as 15, 16, 18, 20, and 16, respectively. The objective is to decide on the number of volunteers that should start at each hour (8:00, 9:00, 10:00, 11:00, and 1:00) such that the given goals are met as much as possible.

C H A P T E R 9

Integer Linear Programming

Integer linear programs (ILPs) are linear programs in which some or all the variables are restricted to integer (or discrete) values. ILP has important practical applications. Unfortunately, despite decades of extensive research, computational experience with ILP has been less than satisfactory. To date, there does not exist an ILP computer code that can solve integer programming problems consistently.

9.1 ILLUSTRATIVE APPLICATIONS

The ILP applications in this section start with simple formulations and then graduate to more complex ones. For convenience, we define a **pure** integer problem as the one in which all the variables are integer. Otherwise, the problem is a **mixed** integer program.

Example 9.1-1 (Capital Budgeting)

Five projects are being evaluated over a 3-year planning horizon. The following table gives the expected returns for each project and the associated yearly expenditures.

	Expenditures (million \$)/yr			
Project	*1*	*2*	*3*	*Returns (million \$)*
1	5	1	8	20
2	4	7	10	40
3	3	9	2	20
4	7	4	1	15
5	8	6	10	30
Available funds (million \$)	25	25	25	

Which projects should be selected over the 3-year horizon?

The problem reduces to a "yes-no" decision for each project. Define the binary variable x_j as

$$x_j = \begin{cases} 1, & \text{if project } j \text{ is selected} \\ 0, & \text{if project } j \text{ is not selected} \end{cases}$$

The ILP model is thus given as

$$\text{Maximize } z = 20x_1 + 40x_2 + 20x_3 + 15x_4 + 30x_5$$

subject to

$$\begin{aligned} 5x_1 + 4x_2 + 3x_3 + 7x_4 + 8x_5 &\le 25 \\ x_1 + 7x_2 + 9x_3 + 4x_4 + 6x_5 &\le 25 \\ 8x_1 + 10x_2 + 2x_3 + x_4 + 10x_5 &\le 25 \\ x_1, x_2, x_3, x_4, x_5 &= (0, 1) \end{aligned}$$

The optimum integer solution (obtained by TORA[1]) is $x_1 = x_2 = x_3 = x_4 = 1$, $x_5 = 0$, with $z = 95$ (million \$). The solution shows that all but project 5 must be selected.

It is interesting to compare the continuous LP solution with the ILP solution. The LP optimum, obtained by replacing $x_j = (0, 1)$ with $0 \le x_j \le 1$ for all j, yields $x_1 = .5789$, $x_2 = x_3 = x_4 = 1$, $x_5 = .7368$, and $z = 108.68$ (million \$). The solution is meaningless because two of the variables assume fractional values. We may *round* the solution to the closest integer values, which yields $x_1 = x_5 = 1$. However, the resulting solution is infeasible because the constraints are violated. More important, the concept of *rounding* should not apply here because x_j represents a "yes-no" decision for which fractional values are meaningless.

PROBLEM SET 9.1A[2]

1. In the capital budgeting model of Example 9.1-1, suppose that project 5 must be selected if either project 1 or project 3 is selected. Modify the model to include the new restriction and find the optimum solution with TORA.
2. Five items are to be loaded in a vessel. The weight w_i and volume v_i together with the value r_i for item i are tabulated below.

Item i	Unit weight, w_i (tons)	Unit volume, v_i (yd^3)	Unit worth, r_i (100 \$)
1	5	1	4
2	8	8	7
3	3	6	6
4	2	5	5
5	7	4	4

The maximum allowable cargo weight and volume are 112 tons and 109 yd^3, respectively. Formulate the ILP model, and find the most valuable cargo using TORA.

[1]To use TORA, select Integer Programming from Main Menu. After inputting the problem (file Ch9ToraCapitalBudgetEx9-1-1.txt), go to output screen and select Automated Run to obtain the optimum solution.
[2]Problems 3 to 6 are adapted from Malba Tahan, *El Hombre Que Calculaba*, Editorial Limusa, Mexico City, 1994, pp. 39–182.

3. Suppose that you have 7 full wine bottles, 7 half-full, and 7 empty. You would like to divide the 21 bottles among three individuals so that each will receive exactly 7. Additionally, each individual must receive the same quantity of wine. Express the problem as an ILP constraint equations, and find a solution using TORA. (*Hint:* Use a dummy objective function in which all the objective coefficients are zeros.)
4. An eccentric sheikh left a will to distribute a herd of camels among his three children: Tarek receives at least one-half of the herd, Sharif gets at least one-third, and Maisa gets at least one-ninth. The remainder goes to a charity organization. The will does not specify the size of the herd except to say that it is an odd number of camels and that the named charity receives exactly one camel. How many camels did the sheikh leave in the estate, and how many did each child get?
5. A farm couple is sending their three children to the market to sell 90 apples with the objective of educating them about money and numbers. Karen, the oldest, carries 50 apples; Bill, the middle child, carries 30; and John, the youngest, carries only 10. The parents have stipulated five rules: (a) The selling price is either \$1 for 7 apples or \$3 for 1 apple, or a combination of the two prices; (b) each child may exercise one or both options of the selling price; (c) each of the three children must return with exactly the same amount of money; (d) each child's income must be in whole dollars (no cents allowed); and (e) the amount received by each child must be the largest possible under the stipulated conditions. Given that the three children are able to sell all they have, how can they satisfy their parents' conditions?
6. Once upon a time, there was a captain of a merchant ship who wanted to reward three crew members for their valiant effort in saving the ship's cargo during an unexpected storm in the high seas. The captain put aside a certain sum of money in the purser's office and instructed the first officer to distribute it equally among the three mariners after the ship had reached shore. One night, one of the sailors, unbeknownst to the others, went to the purser's office and decided to claim (an equitable) one-third of the money in advance. After dividing the money into three equal shares, an extra coin remained, which the mariner decided to keep (in addition to one-third of the money). The next night, the second mariner got the same idea and, repeating the same three-way division with what was left, ended up keeping an extra coin as well. The third night, the third mariner also took a third of what was left, plus an extra coin that could not be divided. When the ship reached shore, the first officer divided what was left of the money equally among the three mariners, also to be left with an extra coin. To simplify things, the first officer put the extra coin aside and gave the three mariners their allotted equal shares. How much money was in the safe to start with? Formulate the problem as an ILP, and find the solution using TORA. (*Hint:* The problem has a countably infinite number of integer solutions. For convenience, assume that we are interested in determining the smallest sum of money that satisfies the problem. Then, boosting the resulting solution by 1, augment it as a lower bound and obtain the next smallest solution. Continuing in this manner, a general solution pattern will evolve.)
7. You have the following three-letter words: AFT, FAR, TVA, ADV, JOE, FIN, OSF, and KEN. Suppose that we assign numeric values to the alphabet starting with $A = 1$ and ending with $Z = 26$. Each word is scored by adding the numeric codes of its three letters. For example, AFT has a score of $1 + 6 + 20 = 27$. You are to select five of the given eight words that yield the maximum total score. Simultaneously, the selected five words must satisfy the following conditions:

$$\begin{pmatrix}\text{sum of letter 1}\\\text{scores}\end{pmatrix} < \begin{pmatrix}\text{sum of letter 2}\\\text{scores}\end{pmatrix} < \begin{pmatrix}\text{sum of letter 3}\\\text{scores}\end{pmatrix}$$

Formulate the problem as an ILP, and find the optimum solution using TORA.

8. The Record-a-Song Company has contracted with a rising star to record eight songs. The durations of the different songs are 8, 3, 5, 5, 9, 6, 7, and 12 minutes, respectively. Record-a-Song uses a two-sided cassette tape for the recording. Each side has a capacity of 30 minutes. The company would like to distribute the songs on the two sides in a balanced manner. This means that the length of the songs on each side should be about the same, as much as possible. Formulate the problem as an ILP, and find the optimum solution.
9. In Problem 8, suppose that the nature of the melodies dictates that songs 3 and 4 cannot be recorded on the same side. Formulate the problem as an ILP. Would it be possible to use a 25-minute tape (each side) to record the eight songs? If not, use ILP to determine the minimum tape capacity needed to make the recording.

Example 9.1-2 (Fixed-Charge Problem)

I have been approached by three telephone companies to subscribe to their long distance service in the United States. MaBell will charge a flat \$16 per month plus \$.25 a minute. PaBell will charge \$25 a month but will reduce the per minute cost to \$.21. As for BabyBell, the flat monthly charge is \$18, and the cost per minute is \$.22. I usually make an average of 200 minutes of long-distance calls a month. Assuming that I do not pay the flat monthly fee unless I make calls and that I can apportion my calls among all three companies as I please, how should I use the three companies to minimize my monthly telephone bill?

This problem can be solved readily without ILP. Nevertheless, it is instructive to formulate it as an integer program.

Define

$$x_1 = \text{MaBell long-distance minutes per month}$$
$$x_2 = \text{PaBell long-distance minutes per month}$$
$$x_3 = \text{BabyBell long-distance minutes per month}$$
$$y_1 = 1 \text{ if } x_1 > 0 \text{ and } 0 \text{ if } x_1 = 0$$
$$y_2 = 1 \text{ if } x_2 > 0 \text{ and } 0 \text{ if } x_2 = 0$$
$$y_3 = 1 \text{ if } x_3 > 0 \text{ and } 0 \text{ if } x_3 = 0$$

We can ensure that y_j will equal 1 if x_j is positive by using the constraint

$$x_j \le My_j,\ j = 1, 2, 3$$

The value of M should be selected sufficiently large so as not to restrict the variables x_j artificially. Because I make about 200 minutes of phone calls a month, then $x_j \le 200$ for all j, and it is safe to select $M = 200$.

The complete model is

$$\text{Minimize } z = .25x_1 + .21x_2 + .22x_3 + 16y_1 + 25y_2 + 18y_3$$

subject to

$$x_1 + x_2 + x_3 \ge 200$$
$$x_1 \le 200y_1$$
$$x_2 \le 200y_2$$

$$x_3 \le 200y_3$$
$$x_1, x_2, x_3 \ge 0$$
$$y_1, y_2, y_3 = (0, 1)$$

The formulation shows that the jth monthly flat fee will be part of the objective function z only if $y_j = 1$, which can happen only if $x_j > 0$ (per the last three constraints of the model). If $x_j = 0$ at the optimum, then the minimization of z, together with the fact that the objective coefficient of y_j is strictly positive, will force y_j to equal zero, as desired.

The optimum solution (file Ch9ToraFixedChargeEx9-1-2.txt) yields $x_3 = 200$, $y_3 = 1$, and all the remaining variables are equal to zero, which shows that BabyBell should be selected as my long-distance carrier. Observe that the information conveyed by $y_3 = 1$ is redundant because the same result is implied by $x_3 > 0$ ($= 200$). Actually, the main reason for using y_1, y_2, and y_3 is to account for the monthly flat fee. In effect, the three binary variables convert an ill-behaved (nonlinear) model into an analytically tractable formulation. This conversion has resulted in introducing the integer (binary) variables in an otherwise continuous problem.

The concept of "flat fee" is typical of what is known in the literature as the **fixed charge problem.**

PROBLEM SET 9.1B

1. Jobco is planning to produce at least 2000 widget on three machines. The minimum lot size on any machine is 500 widget. The following table gives the pertinent data of the situation.

Machine	Setup cost	Production cost/unit	Capacity (units)
1	300	2	600
2	100	10	800
3	200	5	1200

Formulate the problem as an ILP, and find the optimum solution using TORA.

2. Oilco is considering two potential drilling sites for reaching four targets (possible oil wells). The following table provides the preparation costs at each of the two sites and the cost of drilling from site i to target j ($i = 1, 2$; $j = 1, 2, 3, 4$).

	Drilling cost (million \$) to target				
Site	*1*	*2*	*3*	*4*	*Preparation cost (million \$)*
1	2	1	8	5	5
2	4	6	3	1	6

Formulate the problem as an ILP, and find the optimum solution using TORA.

3. Three industrial sites are considered for locating manufacturing plants. The plants send their supplies to three customers. The supply at the plants and the demand at the

customers, together with the unit transportation cost from the plants to the customers, are given in the following table.

	1	2	3	Supply
1	$10	$15	$12	**1800**
2	$17	$14	$20	**1400**
3	$15	$10	$11	**1300**
Demand	**1200**	**1700**	**1600**	

In addition to the transportation costs, fixed costs also are incurred at the rate of $12,000, $11,000, and $12,000 for plants 1, 2, and 3, respectively. Formulate the problem as an ILP and find the optimum solution using TORA.

4. Repeat Problem 3 assuming that the demands at each of customers 2 and 3 are changed to 800.

Example 9.1-3 (Set Covering Problem)

To promote on-campus safety, the U of A Security Department is in the process of installing emergency telephones at selected locations. The department wants to install the minimum number of telephones provided that each of the campus main streets is served by at least one telephone. Figure 9.1 maps the principal streets (A to K) on campus.

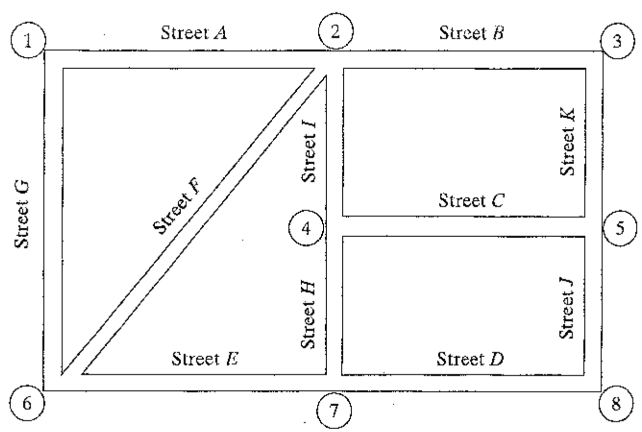
It is logical to place the telephones at the intersections of streets so that each telephone will serve at least two streets. Figure 9.1 shows that the layout of the streets requires a maximum of eight telephone locations.

Define

$$x_j = \begin{cases} 1, & \text{a telephone is installed in location } j \\ 0, & \text{otherwise} \end{cases}$$

FIGURE 9.1

Street map of the U of A campus



The constraints of the problem require installing at least one telephone on each of the 11 streets (A to K). Thus, the model becomes

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$$

subject to

$$\begin{aligned}
x_1 + x_2 &\geq 1 \quad (\text{Street } A)\\
x_2 + x_3 &\geq 1 \quad (\text{Street } B)\\
x_4 + x_5 &\geq 1 \quad (\text{Street } C)\\
x_7 + x_8 &\geq 1 \quad (\text{Street } D)\\
x_6 + x_7 &\geq 1 \quad (\text{Street } E)\\
x_2 + x_6 &\geq 1 \quad (\text{Street } F)\\
x_1 + x_6 &\geq 1 \quad (\text{Street } G)\\
x_4 + x_7 &\geq 1 \quad (\text{Street } H)\\
x_2 + x_4 &\geq 1 \quad (\text{Street } I)\\
x_5 + x_8 &\geq 1 \quad (\text{Street } J)\\
x_3 + x_5 &\geq 1 \quad (\text{Street } K)\\
x_j = (0, 1),\ j &= 1, 2, \ldots, 8
\end{aligned}$$

The optimum solution of the problem (obtained by TORA, file Ch9ToraSetCoverEx9-1-3.txt) requires installing four telephones at intersections 1, 2, 5, and 7. The problem has alternative optima.

The preceding model is typical of what is generically known as the **set covering problem.** In this model, all the variables are binary. For each constraint, all the left-hand-side coefficients are 0 or 1, and the right-hand side is of the form (≥ 1). The objective function always minimizes $c_1x_1 + c_2x_2 + \ldots + c_nx_n$, where $c_j > 0$ for all $j = 1, 2, \ldots, n$. In the present example, $c_j = 1$ for all j. However, if c_j represents the installation cost in location j, then these coefficients may assume values other than 1.

PROBLEM SET 9.1C

1. ABC is an LTL trucking company that delivers loads on a daily basis to five customers. The following table provides the customers associated with each route:

Route	Customers
1	1, 2, 3, 4
2	4, 3, 5
3	1, 2, 5
4	2, 3, 5
5	1, 4, 2
6	1, 3, 5

The segments of each route are dictated by the capacity of the truck delivering the loads. For example, on route 1, the capacity of the truck is sufficient to deliver the loads to customers 1, 2, 3, and 4 only. The following table lists distances (in miles) among the truck terminal (ABC) and the five customers.

	ABC	1	2	3	4	5
ABC	0	10	12	16	9	8
1	10	0	32	8	17	10
2	12	32	0	14	21	20
3	16	8	14	0	15	18
4	9	17	21	15	0	11
5	8	10	20	18	11	0

The objective is to determine the least distance needed to make the daily deliveries to all five customers. Though the solution may result in a customer being served by more than one route, the implementation phase will use only one such route. Formulate the problem as an ILP and solve using TORA.

2. The U of A is in the process of forming a committee to handle the students' grievances. The directive received from the administration is to include at least one female, one male, one student, one administrator, and one faculty member. Ten individuals (identified, for simplicity, by the letters a to j) have been nominated. The mix of these individuals in the different categories is given as follows:

Category	Individuals
Females	a, b, c, d, e
Males	f, g, h, i, j
Students	a, b, c, j
Administrators	e, f
Faculty	d, g, h, i

The U of A wants to form the smallest committee with representation from each of the five categories. Formulate the problem as an ILP, and find the optimum solution using TORA.

3. Washington County includes six towns that need emergency ambulance service. Because of the proximity of some of the towns, a single station may serve more than one community. The stipulation is that the station must be within 15 minutes of driving time from the towns it serves. The table below gives the driving times in minutes among the six towns.

	1	2	3	4	5	6
1	0	23	14	18	10	32
2	23	0	24	13	22	11
3	14	24	0	60	19	20
4	18	13	60	0	55	17
5	10	22	19	55	0	12
6	32	11	20	17	12	0

Formulate an ILP whose solution will produce the smallest number of stations and their locations. Find the solution using TORA.

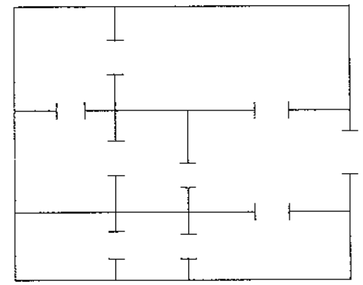


FIGURE 9.2
Museum layout for Problem 4, Set 9.1c

4. The treasures of King Tut are on display in a museum in New Orleans. The layout of the museum is shown in Figure 9.2, with the different rooms joined by open doors. A guard standing at a door can watch two adjoining rooms. The museum wants to ensure guard presence in every room, using the minimum number possible. Formulate the problem as an ILP, and find the optimum solution with TORA.

Example 9.1-4 (Either-or Constraints)

Jobco uses a single machine to process three jobs. Both the processing time and the due date (in days) for each job are given in the following table. The due dates are measured from the zero datum, the assumed start time of the first job.

Job	Processing time (days)	Due date (days)	Late penalty $/day
1	5	25	19
2	20	22	12
3	15	35	34

The objective of the problem is to determine the minimum late-penalty sequence for processing the three jobs.

Define

$$x_j = \text{Start date in days for job } j \text{ (measured from the zero datum)}$$

The problem has two types of constraints: The noninterference constraints (guaranteeing that jobs are not processed concurrently) and the due date constraints. Consider the noninterference constraints first.

Two jobs i and j with processing time p_i and p_j will not be processed concurrently if either $x_i \geq x_j + p_j$ or $x_j \geq x_i + p_i$, depending on whether job j precedes job i, or vice versa. Because all mathematical programs deal with *simultaneous* constraints only, we transform the either-or constraints by introducing the following auxiliary binary variable:

$$y_{ij} = \begin{cases} 1, & \text{if } i \text{ precedes } j \\ 0, & \text{if } j \text{ precedes } i \end{cases}$$

For M sufficiently large, the *either-or* constraint is converted to the following *simultaneous* constraints

$$My_{ij} + (x_i - x_j) \geq p_j \text{ and } M(1 - y_{ij}) + (x_j - x_i) \geq p_i$$

The conversion guarantees that only one of the two constraints can be active at any one time. If $y_{ij} = 0$, the first constraint is active, and the second is redundant (because its left-hand side will include M, which is much larger than p_i). If $y_{ij} = 1$, the first constraint is redundant, and the second is active.

Next, the due date constraint is considered. Given d_j is the due date for job j, let s_j be an unrestricted variable. Then, the associated constraint is

$$x_j + p_j + s_j = d_j$$

If $s_j \geq 0$, the due date is met, and if $s_j < 0$, a late penalty is incurred. Using the substitution

$$s_j = s_j^+ - s_j^-, s_j^+, s_j^- \geq 0$$

the constraint becomes

$$x_j + s_j^+ - s_j^- = d_j - p_j$$

The late penalty cost is proportional to s_j^-.

The model for the given problem is

$$\text{Minimize } z = 19s_1^- + 12s_2^- + 34s_3^-$$

subject to

$$\begin{array}{lllllll}
x_1 - x_2 & + My_{12} & & & & & \geq 20 \\
-x_1 + x_2 & - My_{12} & & & & & \geq 5 - M \\
x_1 - x_3 & & + My_{13} & & & & \geq 15 \\
-x_1 + x_3 & & - My_{13} & & & & \geq 5 - M \\
x_2 - x_3 & & & + My_{23} & & & \geq 15 \\
-x_2 + x_3 & & & - My_{23} & & & \geq 20 - M \\
x_1 & & & & + s_1^+ - s_1^- & & = 25 - 5 \\
x_2 & & & & & + s_2^+ - s_2^- & = 22 - 20 \\
x_3 & & & & & + s_3^+ - s_3^- & = 35 - 15
\end{array}$$

$$x_1, x_2, x_3, s_1^+, s_1^-, s_2^+, s_2^-, s_3^+, s_3^- \geq 0$$

$$y_{12}, y_{13}, y_{23} = (0, 1)$$

The integer variables— y_{12}, y_{13}, and y_{23} —are introduced to convert the either-or constraints into simultaneous constraints. The resulting model is a *mixed* ILP.

To solve the model, we choose $M = 1000$, a value that is larger than the sum of the processing times for all three activities.

The optimal solution (obtained by TORA, file Ch9ToraEitherOrEx9-1-4.txt[3]) is $x_1 = 20$, $x_2, = 0$, and $x_3 = 25$. This means that job 2 starts at time 0, job 1 starts at time 20, and job 3 starts at time 25, thus yielding the optimal processing sequence $2 \rightarrow 1 \rightarrow 3$. The solution calls for completing job 2 at time $0 + 20 = 20$, job 1 at time $= 20 + 5 = 25$, and job 3 at $25 + 15 = 40$ days. Job 3 is delayed by $40 - 35 = 5$ days past its due date at a cost of $5 \times \$34 = \$170$.

[3]Because TORA does not accept a negative right-hand side, the variable (RHS-), whose value is always 1, assumes the role of the right-hand side of the constraints.

PROBLEM SET 9.1D

1. A game board consists of nine equal squares. You are required to fill each square with a number between 1 and 9 such that the sum of the numbers in each row, each column, and each diagonal equals 15. Use ILP to determine the number in each square such that no two adjacent numbers in any row, column, or diagonal are equal. Solve with TORA.
2. A machine is used to produce two interchangeable products. The daily capacity of the machine can produce at most 20 units of product 1 and 10 units of product 2. Alternatively, the machine can be adjusted to produce at most 12 units of product 1 and 22 units of product 2 daily. Market analysis shows that the maximum daily demand for the two products combined is 35 units. Given that the unit profits for the two respective products are \$10 and \$12, which of the two machine settings should be selected? Formulate the problem as an ILP, and find the optimum using TORA. (*Note:* This two-dimensional problem can be solved by inspecting the graphical solution space. This is not the case for the n-dimensional problem.)
3. Gapco manufactures three products, whose daily labor and raw material requirements are given in the following table.

Product	Required daily labor (hr/unit)	Required daily raw material (lb/unit)
1	3	4
2	4	3
3	5	6

The profits per unit of the three products are \$25, \$30, and \$22, respectively. Gapco has two options for locating its plant. The two locations differ primarily in the availability of labor and raw material as shown in the following table:

Location	Available daily labor (hr)	Available daily raw material (lb)
1	100	100
2	90	120

Formulate the problem as a mixed ILP, and use TORA to determine the optimum location of the plant.

4. Consider the job-shop scheduling problem that produces two end products using a single machine. The precedence relationships among the eight operations are summarized in Figure 9.3. Let p_j be the processing time for operations j (=1, 2, ..., n). The due dates, measured from the zero datum, for products 1 and 2, are d_1 and d_2, respectively. An operation, once started, must be completed before another starts. Formulate the problem as a mixed ILP.

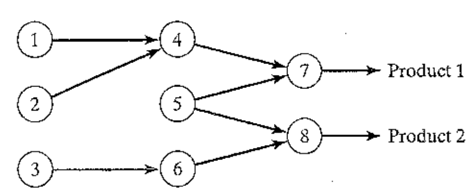


FIGURE 9.3
Precedence relationships for the job-shop situation of Problem 4, Set 9.1d

5. Jaco owns a plant in which three products are manufactured. The labor and raw material requirements for the three products are given in the following table.

Product	Required daily labor (hr/unit)	Required daily raw material (lb/unit)
1	3	4
2	4	3
3	5	6
Daily availability	100	100

The profits per unit for the three products are \$25, \$30, and \$45, respectively. If product 3 is to be manufactured at all, then its production level must be at least 5 units daily. Formulate the problem as a mixed ILP, and find the optimal mix using TORA.

6. Show how the nonconvex shaded solution spaces in Figure 9.4 can be represented by a set of simultaneous constraints. Then use TORA to find the optimum solution that maximizes $z = 2x_1 + 3x_2$ subject to the solution space given in (a).
7. Suppose that it is required that *any* k out of the following m constraints must be active:

$$g_i(x_1, x_2, \ldots, x_n) \leq b_i,\ i = 1, 2, \ldots, m$$

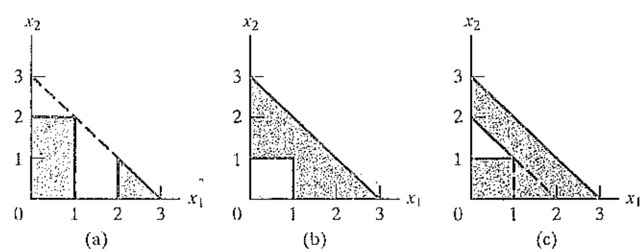
Show how this condition may be represented.

8. In the following constraint, the right-hand side may assume one of the values, $b_1, b_2, \ldots,$ and b_m.

$$g(x_1, x_2, \ldots, x_n) \leq b_1, b_2, \ldots, \text{ or } b_m$$

Show how this condition is represented.

FIGURE 9.4
Solution spaces for Problem 6, Set 9.1d



9.2 INTEGER PROGRAMMING ALGORITHMS

The ILP algorithms are based on exploiting the tremendous computational success of LP. The strategy of these algorithms involves three steps.

Step 1. Relax the solution space of the ILP by deleting the integer restriction on all integer variables and replacing any binary variable y with the continuous range $0 \leq y \leq 1$. The result of the relaxation is a regular LP.

Step 2. Solve the LP, and identify its continuous optimum.

Step 3. Starting from the continuous optimum point, add special constrain iteratively modify the LP solution space in a manner that will even der an optimum extreme point satisfying the integer requirements

Two general methods have been developed for generating the special in step 3.

1. Branch-and-bound (B&B) method
2. Cutting plane method

Although neither method is consistently effective computationally, experi that the B&B method is far more successful than the cutting plane method is discussed further in this chapter.

9.2.1 Branch-and-Bound (B&B) Algorithm

The first B&B algorithm was developed in 1960 by A. Land and G. Doig f eral mixed and pure ILP problem. Later, in 1965, E. Balas developed the ad **rithm** for solving ILP problems with pure binary (zero or one) variables. T algorithm computations were so simple (mainly addition and subtraction) hailed as a possible breakthrough in the solution of general ILP.[4] Unfort algorithm failed to produce the desired computational advantages. Moreove rithm, which initially appeared unrelated to the B&B technique, was shown special case of the general Land and Doig algorithm.

This section will present the general Land-Doig B&B algorithm only. example is used to explain the details.

Example 9.2-1

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 5$$
$$10x_1 + 6x_2 \leq 45$$
$$x_1, x_2 \text{ nonnegative integer}$$

The lattice points (dots) in Figure 9.5 define ILP solution space. The ass problem, LP0, is defined by removing the integer restrictions. Its optimum $x_1 = 3.75$, $x_2 = 1.25$, and $z = 23.75$.

Because the optimum LP0 solution does not satisfy the integer requir B&B algorithm modifies the solution space in a manner that eventually id

[4]A general ILP can be expressed in terms of binary (0-1) variables as follows. Given an inte with a finite upper bound u (i.e., $0 \leq x \leq u$), then

$$x = 2^0y_0 + 2^1y_1 + 2^2y_2 + \ldots + 2^ky_k$$

The variables $y_0, y_1, \ldots,$ and y_k are binary, and the index k is the smallest integer satisfying 2^{k+}

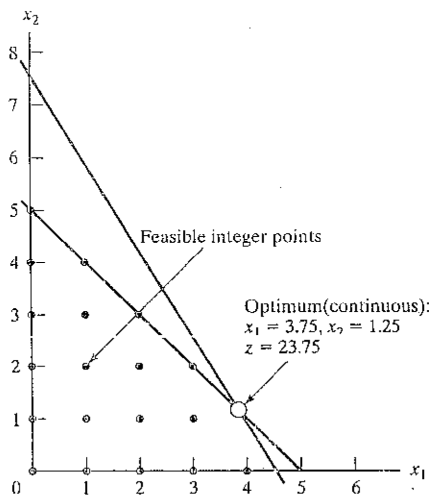


FIGURE 9.5
ILP solution space of Example 9.2-1

ILP optimum. First, we select one of the integer variables whose optimum value at LP0 is not integer. Selecting x_1 (=3.75) arbitrarily, the region $3 < x_1 < 4$ of the LP0 solution space contains no integer values of x_1 and can be eliminated as nonpromising. This is equivalent to replacing the original LP0 with two new LPs, LP1 and LP2, defined as

$$\text{LP1 space} = \text{LP0 space} + (x_1 \le 3)$$

$$\text{LP2 space} = \text{LP0 space} + (x_1 \ge 4)$$

Figure 9.6 depicts the LP1 and LP2 spaces. The two spaces contain the same feasible integer points of the original ILP, which means that, from the standpoint of the integer solution, dealing with LP1 and LP2 is the same as dealing with the original LP0.

If we *intelligently* continue to remove the regions that do not include integer solutions by imposing the appropriate constraints (e.g., $3 < x_1 < 4$ at LP0), we will eventually produce LPs whose optimum extreme points satisfy the integer restrictions. In effect, we will be solving the ILP by dealing with a succession of (continuous) LPs.

The new restrictions, $x_1 \le 3$ and $x_1 \ge 4$, are mutually exclusive, so that LP1 and LP2 must be dealt with as separate LPs as Figure 9.7 shows. This dichotomization gives rise to the concept of **branching** in the B&B algorithm with x_1 being the **branching variable**.

The optimum ILP lies in either LP1 or LP2. Hence, both subproblems must be examined. We arbitrarily examine LP1 (associated with $x_1 \le 3$) first.

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\le 5 \\ 10x_1 + 6x_2 &\le 45 \\ x_1 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

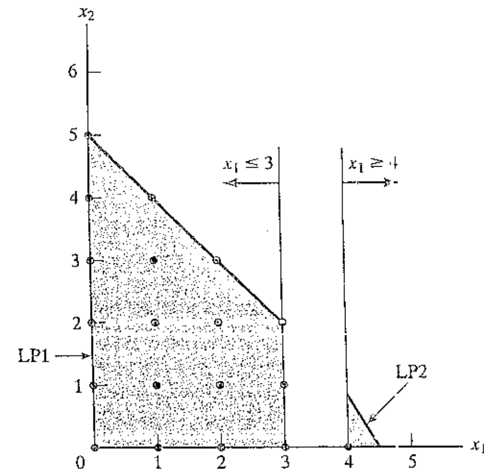


FIGURE 9.6
Solution spaces of LP
for Example 9.2-1

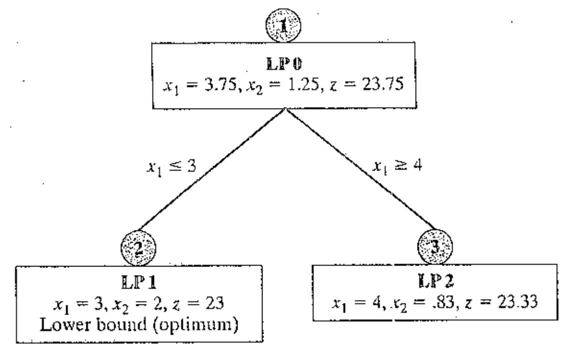


FIGURE 9.7
Using branching variable
ate LP1 and LP2 for Exa

The solution of LP1 (which can be solved efficiently by the upper-bounded a of Section 7.3) yields the optimum solution

$$x_1 = 3, x_2 = 2, \text{ and } z = 23$$

The LP1 solution satisfies the integer requirements for x_1 and x_2. Hence, LP1 be **fathomed**. This means that LP1 need not be investigated any further becau not yield any *better* ILP solution.

We cannot at this point say that the integer solution obtained from LP1 mum for the original problem because LP2 may yield a better integer solutio higher value of z). All we can say is that $z = 23$ is a **lower bound** on the optimu imum) objective value of the original ILP. This means that any unexamined s lem that cannot yield a better objective value than the lower bound must be d as nonpromising. If an unexamined subproblem produces a better integer then the lower bound must be updated accordingly.

Given the lower bound $z = 23$, we examine LP2 (the only remaining unexamined subproblem). Because optimum $z = 23.75$ at LP0 *and all the coefficients of the objective function happen to be integers,* it is impossible that LP2 (which is more restrictive than LP0) will produce a better integer solution. As a result, we discard LP2 and conclude that it has been *fathomed.*

The B&B algorithm is now complete because both LP1 and LP2 have been examined and fathomed (the first for producing an integer solution and the second for showing that it cannot produce a better integer solution). We thus conclude that the optimum ILP solution is the one associated with the lower bound—namely, $x_1 = 3$, $x_2 = 2$, and $z = 23$.

Two questions remain unanswered regarding the procedure:

1. At LP0, could we have selected x_2 as the *branching variable* in place of x_1?
2. When selecting the next subproblem to be examined, could we have solved LP2 first instead of LP1?

The answer to both questions is "yes." However, ensuing computations could differ dramatically. Figure 9.8, in which LP2 is examined first, illustrates this point. The optimum LP2 solution is $x_1 = 4$, $x_2 = .83$, and $z = 23.33$ (verify using TORA LP module). Because x_2 (=.83) is noninteger, LP2 is investigated further by creating subproblems LP3 and LP4 using the branches $x_2 \le 0$ and $x_2 \ge 1$, respectively. This means that

$$\begin{aligned}\text{LP3 space} &= \text{LP2 space} + (x_2 \le 0)\\ &= \text{LP0 space} + (x_1 \ge 4) + (x_2 \le 0)\\ \text{LP4 space} &= \text{LP2 space} + (x_2 \ge 1)\\ &= \text{LP0 space} + (x_1 \ge 4) + (x_2 \ge 1)\end{aligned}$$

We have three "dangling" subproblems that must be examined: LP1, LP3, and LP4. Suppose that we arbitrarily examine LP4 first. LP4 has no solution, and hence it is fathomed. Next, let us examine LP3. The optimum solution is $x_1 = 4.5$, $x_2 = 0$, and $z = 22.5$. The noninteger value of x_1 (=4.5) leads to the two branches $x_1 \le 4$ and $x_1 \ge 5$, and the creation of subproblems LP5 and LP6 from LP3.

$$\text{LP5 space} = \text{LP0 space} + (x_1 \ge 4) + (x_2 \le 0) + (x_1 \le 4) \equiv \text{LP0 space} + (x_1 = 4) + (x_2 \le 0)$$

$$\text{LP6 space} = \text{LP0 space} + (x_1 \ge 4) + (x_2 \le 0) + (x_1 \ge 5) \equiv \text{LP0 space} + (x_1 \ge 5) + (x_2 \le 0)$$

Now, subproblems LP1, LP5, and LP6 remain unexamined. LP6 is fathomed because it has no feasible solution. Next, LP5 has the integer solution ($x_1 = 4$, $x_2 = 0$, $z = 20$) and, hence, yields a lower bound ($z = 20$) on the optimum ILP solution. We are left with subproblem LP1, whose solution yields a better integer ($x_1 = 3$, $x_2 = 2$, $z = 23$). Thus, the lower bound is updated to $z = 23$. Because *all* the subproblems have been fathomed, the optimum solution is associated with the most up-to-date lower bound—namely, $x_1 = 3$, $x_2 = 2$, and $z = 23$.

The solution sequence in Figure 9.8 (LP0 → LP2 → LP4 → LP3 → LP6 → LP5 → LP1) is a worst-case scenario that, nevertheless, may occur in practice. The example points to a principal weakness of the B&B algorithm: How do we select the next subproblem to be examined, and how do we choose its branching variable?

In Figure 9.7, we were lucky to "stumble" upon a good lower bound at the very first subproblem, LP1, thus allowing us to fathom LP2 without further computations and to terminate the B&B search. In essence, we completed the procedure by solving

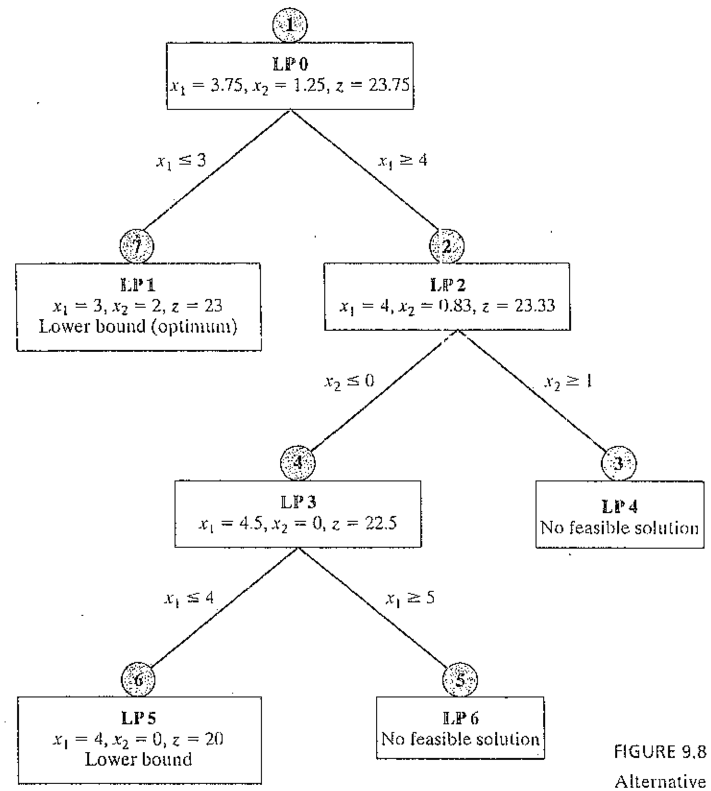


FIGURE 9.8
Alternative B&B tree for Ex

one subproblem only. In Figure 9.8, we had to examine seven subproblems B&B algorithm could be terminated. Although there are heuristics for enl ability of B&B to "guess" which branch can lead to an improved ILP so Taha, 1975, pp. 154–171), there is no solid theory that will always yield results, and herein lies the difficulty that plagues computations in ILP. Section 9.2.2, Problem 1, Set 9.2b, demonstrates with the help of TORA behavior of the B&B algorithm, even for a small 16-variable 1-constrain where the optimum is found in 9 iterations (subproblems) but requires c iterations to verify optimality. It is no wonder that to this day, and after four research, available computer codes (commercial and academic alike) lack ((a la simplex method) in solving ILPs.

We now summarize the B&B algorithm. Assuming a maximization pı an initial lower bound $z = -\infty$ on the optimum objective value of ILP. Set

Step 1. (Fathoming/bounding). Select LP*i*, the next subproblem to be exar Solve LP*i*, and attempt to fathom it using one of three conditions.

(a) The optimal z-value of LPi cannot yield a better objective value than the current lower bound.
(b) LPi yields a better feasible integer solution than the current lower bound.
(c) LPi has no feasible solution.

Two cases will arise.

(a) If LPi is fathomed and a better solution is found, update the lower bound. If *all* subproblems have been fathomed, stop; the optimum ILP is associated with the current lower bound, if any. Otherwise, set $i = i + 1$, and repeat step 1.
(b) If LPi is not fathomed, go to step 2 for branching.

Step 2. (Branching). Select one of the integer variables x_j, whose optimum value x_j^* in the LPi solution is not integer. Eliminate the region

$$[x_j^*] < x_j < [x_j^*] + 1$$

(where $[v]$ defines the largest integer $\leq v$) by creating two LP subproblems that correspond to

$$x_j \leq [x_j^*] \text{ and } x_j \geq [x_j^*] + 1$$

Set $i = i + 1$, and go to step 1.

The given steps apply to maximization problems. For minimization, we replace the lower bound with an upper bound (whose initial value is $z = +\infty$).

The B&B algorithm can be extended directly to mixed problems (in which only some of the variables are integer). If a variable is continuous, we simply never select it as a branching variable. A feasible subproblem provides a new bound on the objective value if the values of the discrete variables are integer and the objective value is improved relative to the current bound.

PROBLEM SET 9.2A

1. Solve the ILP of Example 9.2-1 by the B&B algorithm starting with x_2 as the branching variable. Solve the subproblems with TORA using the MODIFY option for the upper and lower bounds. Start the procedure by solving the subproblem associated with $x_2 \leq [x_2^*]$.
2. Develop the B&B tree for each of the following problems. For convenience, always select x_1 as the branching variable at node 0.

(a) Maximize $z = 3x_1 + 2x_2$

subject to

$$2x_1 + 5x_2 \leq 9$$
$$4x_1 + 2x_2 \leq 9$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

(b) Maximize $z = 2x_1 + 3x_2$

subject to

$$5x_1 + 7x_2 \leq 35$$
$$4x_1 + 9x_2 \leq 36$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

(c) Maximize $z = x_1 + x_2$

subject to

$$2x_1 + 5x_2 \leq 16$$
$$6x_1 + 5x_2 \leq 27$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

(d) Minimize $z = 5x_1 + 4x_2$

subject to

$$3x_1 + 2x_2 \geq 5$$
$$2x_1 + 3x_2 \geq 7$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

(e) Maximize $z = 5x_1 + 7x_2$

subject to

$$2x_1 + x_2 \leq 13$$
$$5x_1 + 9x_2 \leq 41$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

3. Repeat Problem 2, assuming that x_1 is continuous.
4. Show graphically that the following ILP has no feasible solution, and then v result using B&B.

$$\text{Maximize } z = 2x_1 + x_2$$

subject to

$$10x_1 + 10x_2 \leq 9$$
$$10x_1 + 5x_2 \geq 1$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

5. Solve the following problems by B&B.

$$\text{Maximize } z = 18x_1 + 14x_2 + 8x_3 + 4x_4$$

subject to

$$15x_1 + 12x_2 + 7x_3 + 4x_4 + x_5 \leq 37$$
$$x_1, x_2, x_3, x_4, x_5 = (0, 1)$$

9.2.2 TORA-Generated B&B Tree

TORA integer programming module is equipped with a facility for ger B&B tree interactively. To use this facility, select `User-guided B&B` in the ou

8. Convert the following problem into a mixed ILP, and then use TORA to generate its B&B tree. What is the optimal solution?

$$\text{Maximize } z = x_1 + 2x_2 + 5x_3$$

subject to

$$|-x_1 + 10x_2 - 3x_3| \geq 15$$
$$2x_1 + x_2 + x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

9.2.3 Cutting Plane Algorithm

As in the B&B algorithm, the cutting plane algorithm also starts at the continuous optimum LP solution. Special constraints (called **cuts**) are added to the solution space in a manner that renders an optimum integer extreme point. In Example 9.2-2, we first demonstrate graphically how cuts are used to produce an integer solution and then implement the idea algebraically.

Example 9.2-2

Consider the following ILP.

$$\text{Maximize } z = 7x_1 + 10x_2$$

subject to

$$-x_1 + 3x_2 \leq 6$$
$$7x_1 + x_2 \leq 35$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

The cutting plane algorithm modifies the solution space by adding *cuts* that produce an optimum integer extreme point. Figure 9.12 gives an example of two such cuts. Initially, we start with the continuous LP optimum $z = 66\frac{1}{2}$, $x_1 = 4\frac{1}{2}$, $x_2 = 3\frac{1}{2}$. Next, we add cut I, which produces the (continuous) LP optimum solution $z = 62$, $x_1 = 4\frac{4}{7}$, $x_2 = 3$. Then, we add cut II, which, together with cut I and the original constraints, produces the LP optimum $z = 58$, $x_1 = 4$, $x_2 = 3$. The last solution is all desired.

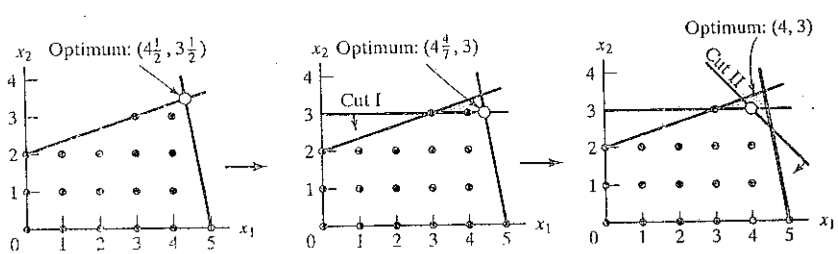


FIGURE 9.12
Illustration of the use of cuts in ILP

The added cuts do not eliminate any of the original feasible integer must pass through at least one feasible or infeasible integer point. These requirements of any cut.

It is purely accidental that a 2-variable problem used exactly 2 cuts to optimum integer solution. In general, the number of cuts, though finite, is in of the size of the problem, in the sense that a problem with a small number c and constraints may require more cuts than a larger problem.

Next, we use the same example to show how the cuts are constructed . mented algebraically.

Given the slacks x_3 and x_4 for constraints 1 and 2, the optimum LP tableau

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$\frac{63}{22}$	$\frac{31}{22}$	$66\frac{1}{2}$
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	$3\frac{1}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	$4\frac{1}{2}$

The optimum continuous solution is $z = 66\frac{1}{2}$, $x_1 = 4\frac{1}{2}$, $x_2 = 3\frac{1}{2}$, $x_3 =$ The cut is developed under the assumption that *all* the variables (including x_3 and x_4) are integer. Note also that because all the original objective coeff integer in this example, the value of z is integer as well.

The information in the optimum tableau can be written explicitly as

$$z + \tfrac{63}{22}x_3 + \tfrac{31}{22}x_4 = 66\tfrac{1}{2} \quad (z\text{-equation})$$
$$x_2 + \tfrac{7}{22}x_3 + \tfrac{1}{22}x_4 = 3\tfrac{1}{2} \quad (x_2\text{-equation})$$
$$x_1 - \tfrac{1}{22}x_3 + \tfrac{3}{22}x_4 = 4\tfrac{1}{2} \quad (x_1\text{-equation})$$

A constraint equation can be used as a **source row** for generating a cut pı right-hand side is fractional. We also note that the z-equation can be used ε row because z happens to be integer in this example. We will demonstrate h generated from each of these source rows, starting with the z-equation.

First, we factor out all the noninteger coefficients of the equation into value and a fractional component, *provided that the resulting fractional cor strictly positive.* For example,

$$\tfrac{5}{2} = (2 + \tfrac{1}{2})$$
$$-\tfrac{7}{3} = (-3 + \tfrac{2}{3})$$

The factoring of the z-equation yields

$$z + (2 + \tfrac{19}{22})x_3 + (1 + \tfrac{9}{22})x_4 = (66 + \tfrac{1}{2})$$

Moving all the integer components to the left-hand side and all the fraction nents to the right-hand side, we get

$$z + 2x_3 + 1x_4 - 66 = -\tfrac{19}{22}x_3 - \tfrac{9}{22}x_4 + \tfrac{1}{2}$$

Because x_3 and x_4 are nonnegative and all fractions are originally strictly pc right-hand side must satisfy the following inequality:

$$-\tfrac{19}{22}x_3 - \tfrac{9}{22}x_4 + \tfrac{1}{2} \leq \tfrac{1}{2}$$

Next, because the left-hand side in Equation (1), $z + 2x_3 + 1x_4 - 66$, is an integer value by construction, the right-hand side, $-\frac{19}{22}x_3 - \frac{9}{22}x_4 + \frac{1}{2}$, must also be integer. It then follows that (2) can be replaced with the inequality:

$$-\tfrac{19}{22}x_3 - \tfrac{9}{22}x_4 + \tfrac{1}{2} \leq 0$$

This is the desired cut, and it represents a necessary condition for obtaining an integer solution. It is also referred to as the **fractional cut** because all its coefficients are fractions.

Because $x_3 = x_4 = 0$ in the continuous LP tableau given above, the current continuous optimum violates the cut (because it yields $\frac{1}{2} \leq 0$). Thus, if we add this cut to the optimum tableau, the resulting optimum extreme point moves the solution toward satisfying the integer requirements.

Before showing how a cut is implemented in the optimal tableau, we will demonstrate how cuts can also be constructed from the constraint equations. Consider the x_1-row:

$$x_1 - \tfrac{1}{22}x_3 + \tfrac{3}{22}x_4 = 4\tfrac{1}{2}$$

Factoring the equation yields

$$x_1 + (-1 + \tfrac{21}{22})x_3 + (0 + \tfrac{3}{22})x_4 = (4 + \tfrac{1}{2})$$

The associated cut is

$$-\tfrac{21}{22}x_2 - \tfrac{3}{22}x_4 + \tfrac{1}{2} \leq 0$$

Similarly, the x_2-equation

$$x_2 + \tfrac{7}{22}x_3 + \tfrac{1}{22}x_4 = 3\tfrac{1}{2}$$

is factored as

$$x_2 + (0 + \tfrac{7}{22})x_3 + (0 + \tfrac{1}{22})x_4 = 3 + \tfrac{1}{2}$$

Hence, the associated cut is given as

$$-\tfrac{7}{22}x_3 - \tfrac{1}{22}x_4 + \tfrac{1}{2} \leq 0$$

Any of the three cuts given above can be used in the first iteration of the cutting plane algorithm. As such, it is not necessary to generate all three cuts before selecting one.

Arbitrarily selecting the cut generated from the x_2-row, we can write it in equation form as

$$-\tfrac{7}{22}x_3 - \tfrac{1}{22}x_4 + s_1 = -\tfrac{1}{2},\ s_1 \geq 0 \qquad \text{(Cut I)}$$

This constraint is added as a secondary constraint to the LP optimum tableau as follows:

Basic	x_1	x_2	x_3	x_4	s_1	Solution
z	0	0	$\frac{63}{22}$	$\frac{31}{22}$	0	$66\frac{1}{2}$
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	0	$3\frac{1}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	0	$4\frac{1}{2}$
s_1	0	0	$-\frac{7}{22}$	$-\frac{1}{22}$	1	$-\frac{1}{2}$

The tableau is optimal but infeasible. We apply the dual simplex method (Section 4.4) to recover feasibility, which yields

Basic	x_1	x_2	x_3	x_4	s_1	Solution
z	0	0	0	1	9	62
x_2	0	1	0	0	1	3
x_1	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	$4\frac{4}{7}$
x_3	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	$1\frac{4}{7}$

The last solution is still noninteger in x_1 and x_3. Let us arbitrarily select next source row—that is,

$$x_1 + (0 + \tfrac{1}{7})x_4 + (-1 + \tfrac{6}{7})s_1 = 4 + \tfrac{4}{7}$$

The associated cut is

$$-\tfrac{1}{7}x_4 - \tfrac{6}{7}s_1 + s_2 = -\tfrac{4}{7},\ s_2 \geq 0 \qquad \text{(Cut II)}$$

Basic	x_1	x_2	x_3	x_4	s_1	s_2	Solution
z	0	0	0	1	9	0	62
x_2	0	1	0	0	1	0	3
x_1	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	0	$4\frac{4}{7}$
x_3	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	0	$1\frac{4}{7}$
s_2	0	0	0	$-\frac{1}{7}$	$-\frac{6}{7}$	1	$-\frac{4}{7}$

The dual simplex method yields the following tableau:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	Solution
z	0	0	0	0	3	7	58
x_2	0	1	0	0	1	0	3
x_1	1	0	0	0	-1	1	4
x_3	0	0	1	0	-4	1	1
x_4	0	0	0	1	6	-7	4

The optimum solution ($x_1 = 4$, $x_2 = 3$, $z = 58$) is all integer. It is not a that all the coefficients of the last tableau are integers. This is a typical prope implementation of the fractional cut.

It is important to point out that the fractional cut assumes that *all* the v *including slack and surplus*, are integer. This means that the cut deals with pur problems only. The importance of this assumption is illustrated by an example

Consider the constraint

$$x_1 + \tfrac{1}{3}x_2 \leq \tfrac{13}{2}$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

From the standpoint of solving the associated ILP, the constraint is treated as tion by using the nonnegative slack s_1—that is,

$$x_1 + \tfrac{1}{3}x_2 + s_1 = \tfrac{13}{2}$$

The application of the fractional cut assumes that the constraint has a feasible integer solution in all x_1, x_2 and s_1. However, the equation above will have a feasible integer solution in x_1 and x_2 *only if s_1 is noninteger.* This means that cutting-plane algorithm will show that the problem has no feasible integer solution, even though the variables of concern, x_1 and x_2, can assume integer feasible values.

There are two ways to remedy this situation.

1. Multiply the entire constraint by a proper constant to remove all the fractions. For example, multiplying the constraint above by 6, we get

$$6x_1 + 2x_2 \leq 39$$

Any integer solution of x_1 and x_2 automatically yields integer slack. However, this type of conversion is appropriate for only simple constraints because the magnitudes of the integer coefficients may become excessively large in some cases.

2. Use a special cut, called the **mixed cut**, which allows only a subset of variables to assume integer values, with all the other variables (including slack and surplus) remaining continuous. The details of this cut will not be presented in this chapter (see Taha, 1975, pp. 198–202).

PROBLEM SET 9.2C

1. In Example 9.2-2, show graphically whether or not each of the following constraints can form a legitimate cut:
   (a) $x_1 + 2x_2 \leq 10$
   (b) $2x_1 + x_2 \leq 10$
   (c) $3x_2 \leq 10$
   (d) $3x_1 + x_2 \leq 15$
2. In Example 9.2-2, show graphically how the following two (legitimate) cuts can lead to the optimum integer solution:

$$x_1 + 2x_2 \leq 10 \qquad \text{(Cut I)}$$

$$3x_1 + x_2 \leq 15 \qquad \text{(Cut II)}$$

3. Express cuts I and II of Example 9.2-2 in terms of x_1 and x_2, and show that they are the same ones used graphically in Figure 9.12.
4. In Example 9.2-2, derive cut II from the x_3-row of the tableau resulting from the application of cut I. Use the new cut to complete the solution of the example.
5. Show that, even though the following problem has a feasible integer solution in x_1 and x_2, the fractional cut would not yield a feasible solution unless all the fractions in the constraint have been eliminated.

$$\text{Maximize } z = x_1 + 2x_2$$

subject to

$$x_1 + \tfrac{1}{2}x_2 \leq \tfrac{13}{4}$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

6. Solve the following problems by the fractional cut, and compare the true optir ger solution with the solution obtained by rounding the continuous optimum.
   (a) Maximize $z = 4x_1 + 6x_2 + 2x_3$
   subject to

$$\begin{aligned} 4x_1 - 4x_2 &\leq 5 \\ -x_1 + 6x_2 &\leq 5 \\ -x_1 + x_2 + x_3 &\leq 5 \\ x_1, x_2, x_3 &\geq 0 \text{ and integer} \end{aligned}$$

   (b) Maximize $z = 3x_1 + x_2 + 3x_3$
   subject to

$$\begin{aligned} -x_1 + 2x_2 + x_3 &\leq 4 \\ 4x_2 - 3x_3 &\leq 2 \\ x_1 - 3x_2 + 2x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \text{ and integer} \end{aligned}$$

9.2.4 Computational Considerations in ILP

To date, and despite over 40 years of research, there does not exist a comp that can solve ILP consistently. Nevertheless, of the two solution algorithms in this chapter, B&B is more reliable. Indeed, practically all commercial are B&B-based. Cutting plane methods are generally difficult and uncertai roundoff error presents a serious problem. Though attempts have been improve the cutting plane computational efficacy, the end results are not en In most cases, the cutting plane method is used in a secondary capacity t B&B performance at each subproblem.

The most important factor affecting computations in integer programr number of integer variables and the feasible range in which they apply. Bec able algorithms are not consistent in producing a numeric ILP solution, advantageous computationally to reduce the number of integer variables i model as much as possible. The following suggestions may prove helpful:

1. Approximate integer variables by continuous ones wherever possible.
2. For the integer variables, restrict their feasible ranges as much as possil
3. Avoid the use of nonlinearity in the model.

The importance of the integer problem in practice is not yet matched l solution algorithms. It is unlikely that a new theoretical breakthrough will b in the area of integer programming. Instead, new technological advances in (such as parallel processing) may offer the best hope for improving the efl ILP codes.

作業研究

LINDO (Linear, Interactive, Discrete Optimizer)

LINDO 是以自然的型式輸入，可以解線性數，二次和整數規劃

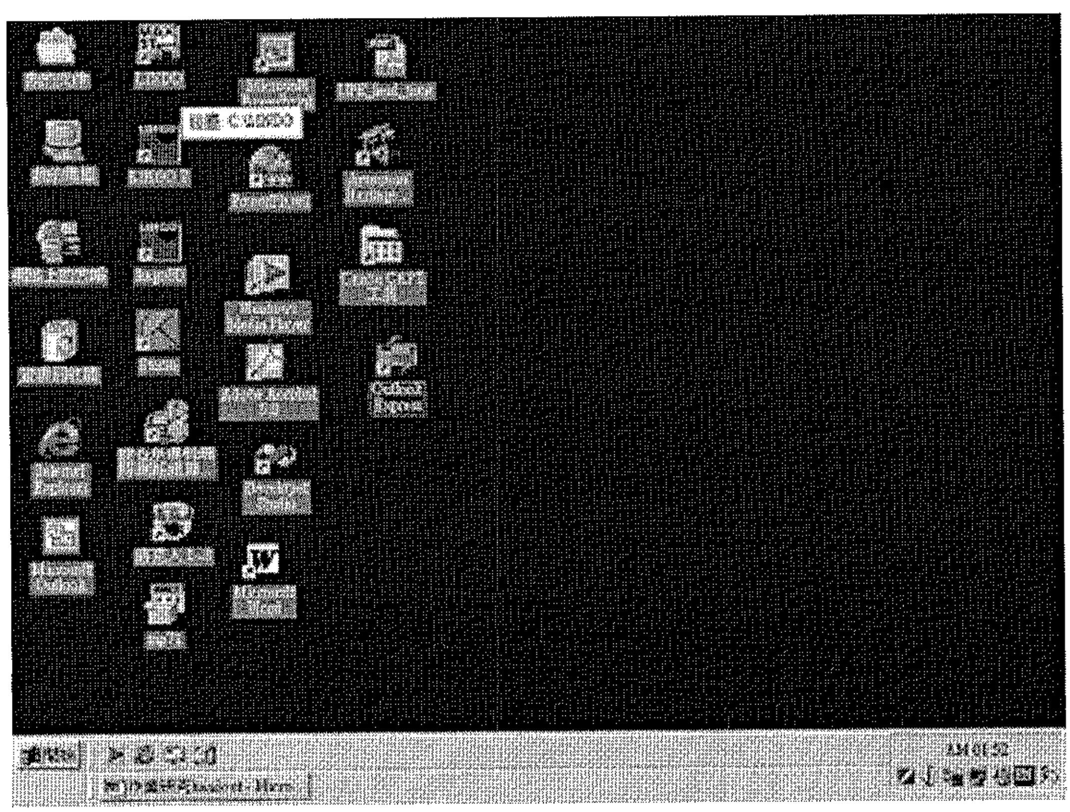
學生版 LINDO 最大輸入	更專業版 LINDO
非零 9758	LINDO SYSTEMS, INC
欄 201	P.O. BOX 148231
列 101	CHICAGO, IL 60614
整數變數 200	U.S.A
整數/列名稱 8	

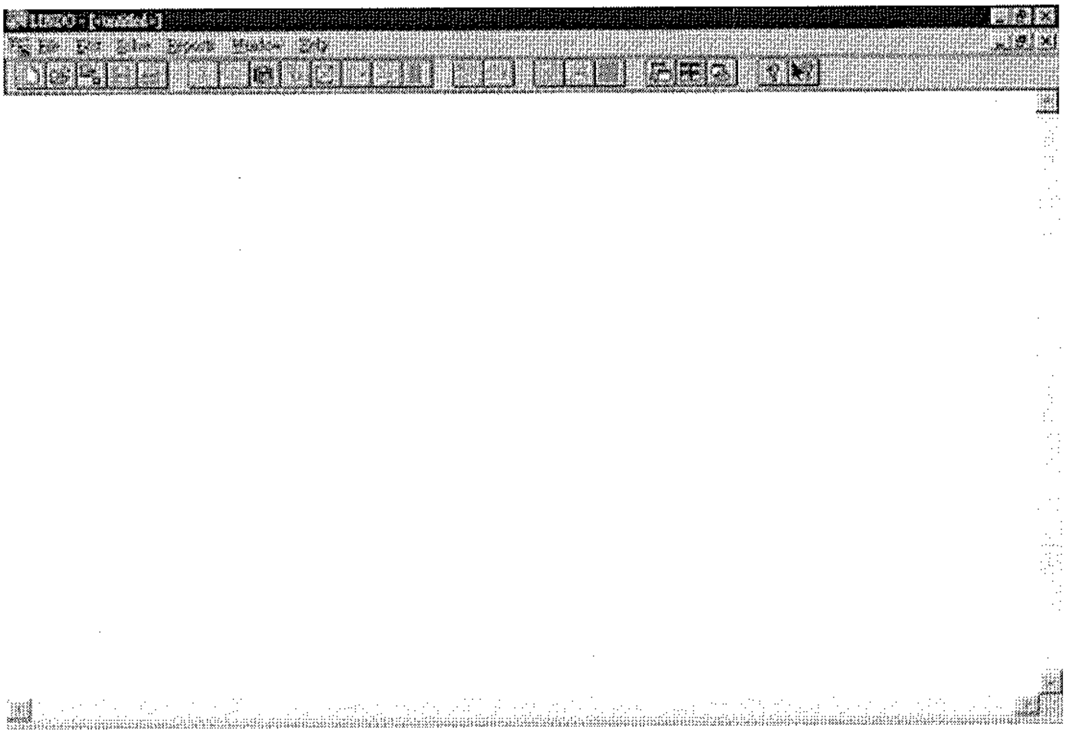
題目 PAGE 18

$$\begin{aligned} \text{MAX} \quad & Z = 3X_E + 2X_I \\ \text{ST} \quad & X_E + 2X_I \leq 6 \\ & 2X_E + 2X_I \leq 8 \\ & -X_E + X_I \leq 1 \\ & X_I \leq 2 \\ & X_E, X_I \geq 0 \end{aligned}$$

輸入方式及解題

點選 LINDO 圖示則進入 LINDO





在 LINDO 的環境輸入如下

max　3XE+2XI

st

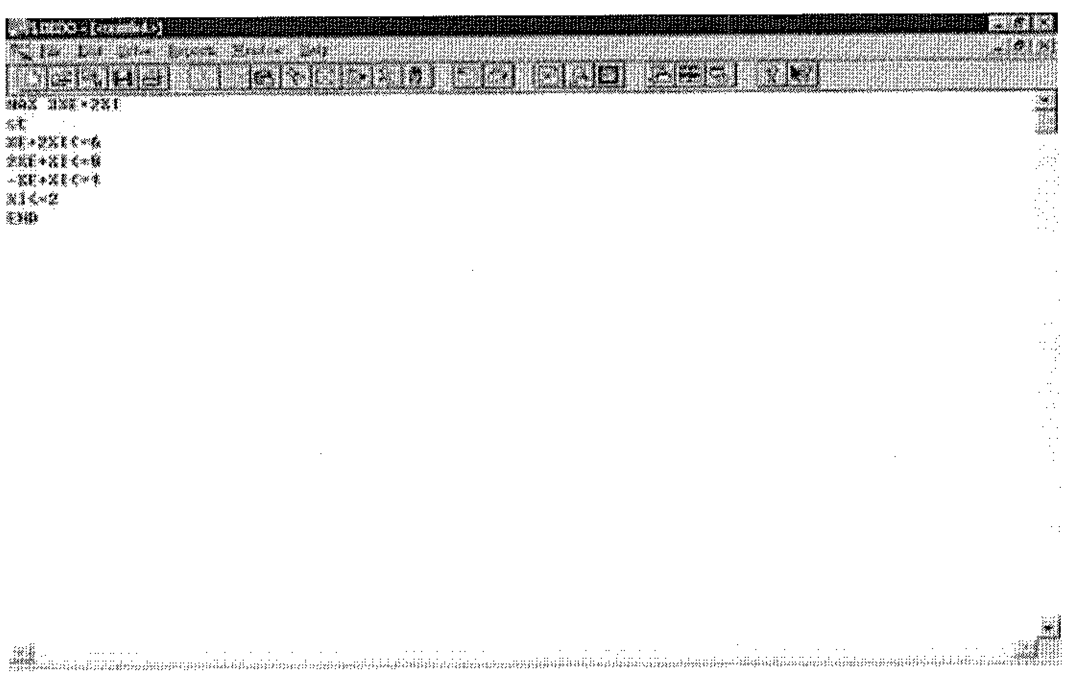
XE+2XI<=6

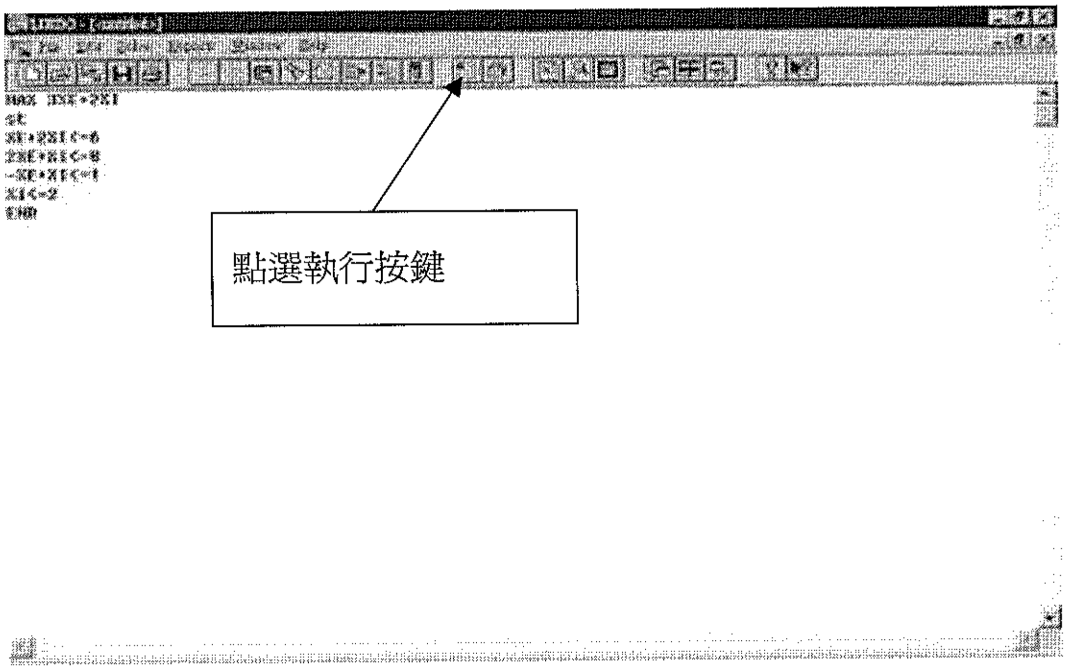
2XE+XI<=8

-XE+XI<=1

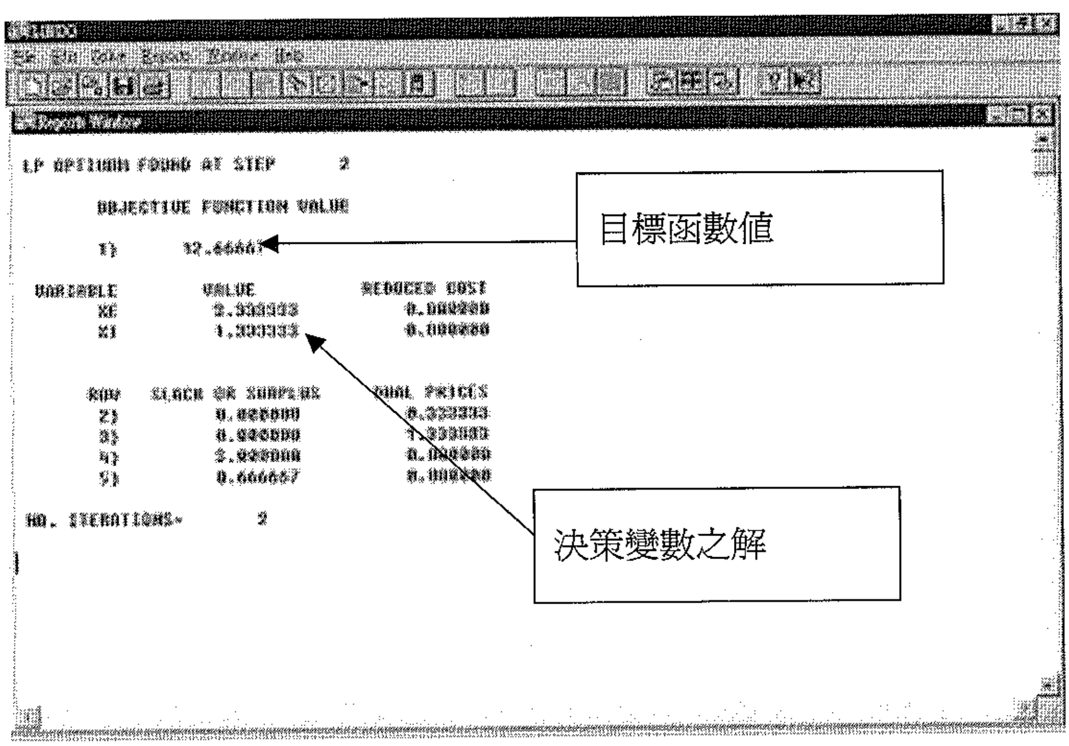
XI<=2

END





如果要做敏感度分析則點選是，否則關閉其他視窗，則產生下列結果



題目)

$$\begin{aligned} \text{Min}\quad & Z = 4X_1 + 2X_2 \\ \text{ST}\quad & 3X_1 + X_2 = 3 \\ & 4X_1 + 3X_2 \geq 6 \\ & X_1 + 2X_2 \leq 4 \\ & X_1,\ X_2 \geq 0 \end{aligned}$$

輸入如下

```
min   4X1+X2
st
3X1+X2=3
4X1+3X2>=6
X1+2X2<=4
END
```

執行方式與上例相同

關於指令更詳細的用法，請使用 help 或參考

Schrage, L. *User's Manual for LINDO*. Palo Alto, Calif:Scientific Press, 1990

例題一人員放假問題

```
! A small staff scheduling model:
! Each employee works a 5 day day shift with two days off.
! A worker can be started any day of the week.  A worker earns
! $100 per week.
! X<day> = Number of  of employees we start on day <day>
MIN      100 XMON + 100 XTUE + 100 XWED + 100 XTHU + 100 XFRI
       + 100 XSAT + 100 XSUN
 SUBJECT TO
                 XWED + XTHU + XFRI + XSAT + XSUN >=    18
  XMON                  + XTHU + XFRI + XSAT + XSUN >=   16
  XMON + XTUE                  + XFRI + XSAT + XSUN >=   15
  XMON + XTUE + XWED                  + XSAT + XSUN >=   16
  XMON + XTUE + XWED + XTHU                  + XSUN >=   19
  XMON + XTUE + XWED + XTHU + XFRI                  >=   14
         XTUE + XWED + XTHU + XFRI + XSAT           >=   12
 END
! Solve the model. The objective should be $2200.
```

例題二運輸問題

```
! A 3 warehouse, 4 customer transportation model:
! XWH<i>C<j> = amount shipped from warehouse <i> to customer
<j>
MIN     6 XWH1C1 + 2 XWH1C2 + 6 XWH1C3 + 7 XWH1C4
     +  4 XWH2C1 + 9 XWH2C2 + 5 XWH2C3 + 3 XWH2C4
     +  8 XWH3C1 + 8 XWH3C2 +   XWH3C3 + 5 XWH3C4
 SUBJECT TO
!  Demand constraints:
  XWH1C1 + XWH2C1 + XWH3C1 >=   15
  XWH1C2 + XWH2C2 + XWH3C2 >=   17
  XWH1C3 + XWH2C3 + XWH3C3 >=   22
  XWH1C4 + XWH2C4 + XWH3C4 >=   12
!  Supply constraints:
  XWH1C1 + XWH1C2 + XWH1C3 + XWH1C4 <=   30
  XWH2C1 + XWH2C2 + XWH2C3 + XWH2C4 <=   25
  XWH3C1 + XWH3C2 + XWH3C3 + XWH3C4 <=   21
 END
! Solve the model. The objective should be 161.
```

例題三載貨問題

```
!  A shipping company wants to load a shipping container to maxmimize
!  the freight charges it can bill.  There is  a cubic space constraint
!  of 1000 sq ft, and a weight limit of 1200 pounds.
!    X<i> = 1 if parcel <i> in included in the container, else 0.
MAX 77 X1 + 6 X2 + 3 X3 + 6 X4 + 33 X5 + 13 X6 + 110 X7 + 21 X8 + 47 X9
SUBJECT TO
  774 X1 + 76 X2 +  22 X3 + 42 X4 + 21 X5 + 760 X6
      + 818 X7 + 62 X8 + 785 X9                          <=    1000
  67 X1 + 27 X2 + 794 X3 + 53 X4 + 234 X5 + 32 X6
      + 792 X7 + 97 X8 + 435 X9                          <=    1200
END
INT 9
! The best integer solution should have  an objective value of 170.
```

例題四選曲問題

```
! We want to decide how to place 7 songs on a record album so as to
! maximize the number of songs on the "short" side of the album.  The
! short side must contain no more than half the total music time. The
! times by song are:
!
!     SONG:    1   2   3   4   5   6   7
!     TIME:    2   5   2   2   7   2   2
!
! Y<i> = 1 if song <i> is assigned to short side, else 1
!
 MAX    Y1 +   Y2 +   Y3 +   Y4 +   Y5 +   Y6 +   Y7
 SUBJECT TO
      2 Y1 + 5 Y2 + 2 Y3 + 2 Y4 + 7 Y5 + 2 Y6 + 2 Y7 <= 11
 END
!
! The Y's must be 0/1:
INT 7
!INT 7 代表前面 7 個變數均爲 0/1 整數
!如果前面 7 個變數並非全爲 0/1 整數，則需各別宣告
!例如 Y1  Y2  Y5  Y6  Y7 爲 0/1 整數，Y3  Y4
!爲一般整數，則宣告如下
!INTE Y1
!INTE Y2
!INTE Y5
!INTE Y6
!INTE Y7
!GIN Y3
!GIN Y4

! Songs 1, 3, 4, 6, and 7 should appear on the short side.
```

$$
\begin{array}{llll}
\min & y_{LP} + y_{IP} + y_{NLP} & & \\
\text{s.t.} & x_1 + x_2 + x_3 + x_4 + y_{LP} & \geq 1 & \text{(LP)} \\
& x_2 + x_4 + y_{IP} & \geq 1 & \text{(IP)} \\
& x_3 + x_4 + y_{NLP} & \geq 1 & \text{(NLP)} \\
& 3x_1 + 4x_2 + 6x_3 + 14x_4 & \leq 12 & \text{(budget)} \\
& x_1, \ldots, x_4 = 0 \text{ or } 1 & & \\
& y_{LP}, y_{IP}, y_{NLP} = 0 \text{ or } 1 & &
\end{array}
$$

:olumn Generation Models

\nother common family of set packing, covering, and partitioning models are de-ived from problems involving a combinatorially large array of possibilities too com-lex to be modeled concisely. Column generation adopt a two-part strategy for such roblems.

11.13 **Column generation** approaches deal with complex combinatorial problems by first enumerating a sequence of columns representing viable solutions to parts of the problem, and then solving a set partitioning (or covering or packing) model to select an optimal collection of these alternatives fulfilling all problem requirements.

The convenience of this approach comes from its flexibility. Any appropriate cheme—however ad hoc—can be employed to generate a rich family of columns neeting complex and difficult-to-model constraints. Optimization is reserved for he second part of the strategy, when a well-formed model over the columns can be uddressed by standard ILP technology.

:XAMPLE 11.4: AA CREW SCHEDULING

\ classic application of column generation arises in the enormously complex problem of scheduling crews for airlines. For example, American Airlines[3] reports spending over \$1.3 billion per year on salaries, benefits, and travel expenses of air crews. Careful scheduling, or **crew pairing** as it is called, can produce enormous savings.

Figure 11.2 illustrates a (tiny) sequence of flights to be crewed in our fictitious :ase. For example, flight 101 originates in Miami and arrives in Chicago some hours ater.

Each pairing is a sequence of flights to be covered by a single crew over a 2- to 3-day period. It must begin and end in the base city where the crew resides.

Table 11.3 enumerates possible pairings of flights in Figure 11.2. For example, pairing $j = 1$ begins at Miami with flight 101. After a layover in Chicago, the crew hen covers flight 203 to Dallas–Ft. Worth and then flight 406 to Charlotte. Finally, light 308 returns them to Miami.

In real applications, intricate government and union rules regulate exactly which sequences of flights constitute a reasonable pairing. Complex software is employed to generate a list such as that of Table 11.3. Here we simply allow all closed sequences of 3 or 4 flights in Figure 11.2.

[3]Based on R. Anbil, E. Gelman, B. Patty, and R. Tanga (1991), "Recent Advances in Crew-Pairing Optimization at American Airlines," *Interfaces 21:1*, 62–74.

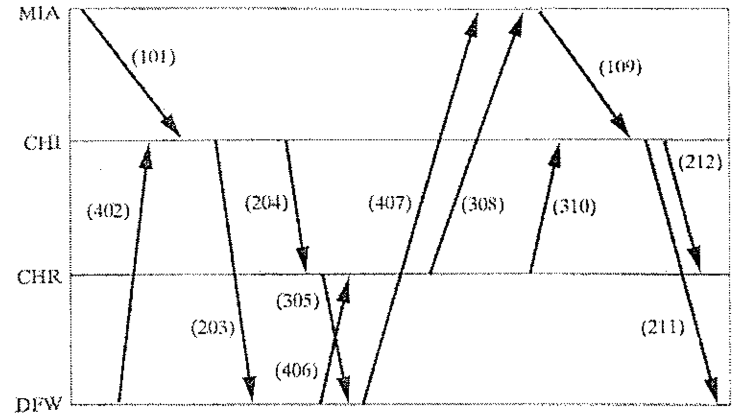


FIGURE 11.2 Flight Schedule for AA Example

TABLE 11.3 Possible Pairings for AA Example

j	Flight Sequence	Cost	j	Flight Sequence	Cost
1	101–203–406–308	2900	9	305–407–109–212	2600
2	101–203–407	2700	10	308–109–212	2050
3	101–204–305–407	2600	11	402–204–305	2400
4	101–204–308	3000	12	402–204–310–211	3600
5	203–406–310	2600	13	406–308–109–211	2550
6	203–407–109	3150	14	406–310–211	2650
7	204–305–407–109	2550	15	407–109–211	2350
8	204–308–109	2500			

Pairing costs are equally complex. On duty crews are guaranteed pay for minimum periods, regardless of the part of that time they are actually flying. If the pairing requires overnight stays away from home, hotel and other expenses are added. Values for our illustration are shown in Table 11.3.

Column Generation Model for AA Example

Having enumerated a list of alternatives like Table 11.3, the remaining task is to find a minimum total cost collection of columns staffing each flight exactly once. Define decision variables

$$x_j \triangleq \begin{cases} 1 & \text{if pairing } j \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$$

Then the following set partitioning model does the job:

$$
\begin{array}{llll}
\min & 2900x_1 + 2700x_2 + 2600x_3 + 3000x_4 + 2600x_5 & & \\
& + 3150x_6 + 2550x_7 + 2500x_8 + 2600x_9 + 2050x_{10} & & \\
& + 2400x_{11} + 3600x_{12} + 2550x_{13} + 2650x_{14} + 2350x_{15} & & \\
\text{s.t.} & x_1 + x_2 + x_3 + x_4 & = 1 & \text{(flight 101)} \\
& x_6 + x_7 + x_8 + x_9 + x_{10} + x_{13} + x_{15} & = 1 & \text{(flight 109)} \\
& x_1 + x_2 + x_5 + x_6 & = 1 & \text{(flight 203)}
\end{array}
\qquad (11.10)
$$

$$
\begin{array}{lll}
x_3 + x_4 + x_7 + x_8 + x_{11} + x_{12} = 1 & & \text{(flight 204)} \\
x_{12} + x_{13} + x_{14} + x_{15} & = 1 & \text{(flight 211)} \\
x_9 + x_{10} & = 1 & \text{(flight 212)} \\
x_3 + x_7 + x_9 + x_{11} & = 1 & \text{(flight 305)} \\
x_1 + x_4 + x_8 + x_{10} + x_{13} & = 1 & \text{(flight 308)} \\
x_5 + x_{12} + x_{14} & = 1 & \text{(flight 310)} \\
x_{11} + x_{12} & = 1 & \text{(flight 402)} \\
x_1 + x_5 + x_{13} + x_{14} & = 1 & \text{(flight 406)} \\
x_2 + x_3 + x_6 + x_7 + x_9 + x_{15} & = 1 & \text{(flight 407)} \\
x_1, \ldots, x_{15} = 0 \text{ or } 1 & &
\end{array}
$$

An optimal solution makes

$$x_1^* + x_9^* + x_{12}^* = 1$$

and all other $x_j^* = 0$ at total cost \$9100.

Sample Exercise 11.9: Forming Column Generation Models

A moving and storage company is allocating 5 long-distance moving loads to trucks. One feasible combination covers loads 1 and 3 in a 4525-mile route; a second combines loads 2, 3, and 4 in 2960 miles; a third hauls loads 2, 4, and 5 in 3170 miles; and a fourth covers load 1, 4, and 5 in 5230 miles. Form a set partitioning model to decide a minimum distance combination of routes covering each load exactly once.

Modeling: We use the decision variables

$$x_j \triangleq \begin{cases} 1 & \text{if route } j \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$$

Then the required model is

$$
\begin{array}{llll}
\min & 4525x_1 + 2960x_2 + 3170x_3 + 5230x_4 & & \text{(total miles)} \\
\text{s.t.} & x_1 + x_4 & = 1 & \text{(load 1)} \\
& x_2 + x_3 & = 1 & \text{(load 2)} \\
& x_1 + x_2 & = 1 & \text{(load 3)} \\
& x_2 + x_3 + x_4 & = 1 & \text{(load 4)} \\
& x_3 + x_4 & = 1 & \text{(load 5)} \\
& x_1, \ldots, x_4 = 0 \text{ or } 1 & &
\end{array}
$$

11.4 Assignment and Matching Models

We have already encountered **assignment problems** in network flow Section 10.5. The issue is optimal matching or pairing of objects of two distinct types—jobs to three events, and at least four of the team participants must be all-rounders. Set up an ILP model that can be used to select the competing team, and find the optimum solution using TORA.

9.3[6] In 1990, approximately 180,000 telemarketing centers employing 2 million individuals were in operation in the United States. In the year 2000, more than 700,000 companies employed approximately 8 million people in telemarketing their products. The questions of how many telemarketing centers to employ and where to locate them are of paramount importance.

The ABC company is in the process of deciding on the number of telemarketing centers to employ and their locations. A center may be located in one of several candidate areas selected by the company and may serve (partially or completely) one or more geographical areas. A geographical area is usually identified by one or more (telephone) area codes. ABC's telemarketing concentrates on eight area codes: 501, 918, 316, 417, 314, 816, 502, and 606. The following table provides the candidate locations, their served areas, and the cost of establishing the center.

Center location	Served area codes	Cost ($)
Dallas, TX	501, 918, 316, 417	500,000
Atlanta, GA	314, 816, 502, 606	800,000
Louisville, KY	918, 316, 417, 314, 816	400,000
Denver, CO	501, 502, 606	900,000
Little Rock, AR	417, 314, 816, 502	300,000
Memphis, TN	606, 501, 316, 417	450,000
St. Louis, MO	816, 502, 606, 314	550,000

Customers in all area codes can access any of the centers 24 hours a day.

The communication costs per hour between the centers and the area codes are given in the following table.

	From area code							
To	*501*	*918*	*316*	*417*	*314*	*816*	*502*	*606*
Dallas, TX	$14	$35	$29	$32	$25	$13	$14	$20
Atlanta, GA	$18	$18	$22	$18	$26	$23	$12	$15
Louisville, KY	$22	$25	$12	$19	$30	$17	$26	$25
Denver, CO	$24	$30	$19	$14	$12	$16	$18	$30
Little Rock, AR	$19	$20	$23	$16	$23	$11	$28	$12
Memphis, TN	$23	$21	$17	$21	$20	$23	$20	$10
St. Louis, MO	$17	$18	$12	$10	$19	$22	$16	$22

ABC would like to select between three and four centers. Where should they be located?