(b) Use hand computations to carry out Phase II with the zero-artificial variable as part of the starting basic solution. Make sure that the artificial variables never assume positive values.

(c) Show that the zero-artificial variable can be driven out of the optimum basic solution of Phase I (before we start Phase II) by selecting an entering variable with a *nonzero* pivot element in the artificial variable row. Then carry out Phase II using the new basic solution.

6. Consider the following problem:

$$\text{Maximize } z = 3x_1 + 2x_2 + 3x_3$$

subject to

$$2x_1 + x_2 + x_3 = 2$$
$$x_1 + 3x_2 + x_3 = 6$$
$$3x_1 + 4x_2 + 2x_3 = 8$$
$$x_1, x_2, x_3 \geq 0$$

(a) Use TORA to show that Phase I terminates with two zero-artificial variables in the basic solution.

(b) Show that when the procedure of Problem 5(c) is applied at the end of Phase I, only one of the two zero-artificial variables can be made nonbasic.

(c) Show that the original constraint associated with the zero-artificial variable that cannot be made nonbasic in (b) must be redundant—hence, its row as well as the artificial variable itself can be dropped altogether at the start of Phase II.

7. Consider the following LP:

$$\text{Maximize } z = 3x_1 + 2x_2 + 3x_3$$

subject to

$$2x_1 + x_2 + x_3 \leq 2$$
$$3x_1 + 4x_2 + 2x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

The optimal simplex tableau at the end of Phase I is given as

Basic	x_1	x_2	x_3	x_4	x_5	R	Solution
z	−5	0	−2	−1	−4	0	0
x_2	2	1	1	0	1	0	2
R	−5	0	−2	−1	−4	1	0

Show that the nonbasic variables $x_1, x_3, x_4,$ and x_5 can never assume positive values at the end of Phase II. Hence, their columns can dropped before we start Phase II. In essence, the removal of these variables reduces the constraint equations of the problem to $x_2 = 2$. This means that it will not be necessary to carry out Phase II at all because the solution space is reduced to one point only.

The general conclusion from this problem is that any nonbasic variables that have *strictly negative* z-row coefficients at the end of Phase I must be dropped from the tableau as they can never assume positive values at the end of Phase II. Incidentally, negative z-row coefficients for nonartificial variables can only occur if an artificial variable is basic (at zero level) at the end of Phase I.

8. Consider the LP model

$$\text{Minimize } z = 2x_1 - 4x_2 + 3x_3$$

subject to

$$5x_1 - 6x_2 + 2x_3 \geq 5$$
$$-x_1 + 3x_2 + 5x_3 \geq 8$$
$$2x_1 + 5x_2 - 4x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

Show how the inequalities can be modified to a set of equations that requires the use of single artificial variable only (instead of two).

3.5 SPECIAL CASES IN SIMPLEX METHOD APPLICATION

This section considers four special cases that arise in the application of the simplex method.

1. Degeneracy
2. Alternative optima
3. Unbounded solutions
4. Nonexisting (or infeasible) solutions

Our interest in studying these special cases is twofold: (1) to present a *theoretical* explanation for the reason these situations arise and (2) to provide a *practical* interpretation of what these special results could mean in a real-life problem.

3.5.1 Degeneracy

In the application of the feasibility condition of the simplex method a tie for the minimum ration may be broken arbitrarily for the purpose of determining the leaving variable. When this happens, one or more of the *basic* variables will be zero in the next iteration. In this case, the new solution is **degenerate**.

There is nothing alarming about dealing with a degenerate solution, with the exception of a small theoretical inconvenience, which we shall discuss shortly. From the practical standpoint, the condition reveals that the model has at least one *redundant* constraint. To be able to provide more insight into the practical and theoretical impacts of degeneracy, we consider a numeric example. The graphical illustration should enhance the understanding of ideas underlying this special situation.

Example 3.5-1 (DEGENERATE OPTIMAL SOLUTION).

$$\text{Maximize } z = 3x_1 + 9x_2$$

subject to

$$x_1 + 4x_2 \le 8$$

$$x_1 + 2x_2 \le 4$$

$$x_1, x_2 \ge 0$$

Let x_3 and x_4 be slack variables. The simplex iterations are given in the following tableux.

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	-3	-9	0	0	0
x_2 enters	x_3	1	4	1	0	8
x_3 leaves	x_4	1	2	0	1	4
1	z	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18
x_1 enters	x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0
2	z	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18
(optimum)	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2
	x_1	1	0	-1	2	0

In the starting iteration, x_3 and x_4 tie for the leaving variable. This is the reason the basic variable x_4 has a zero value in iteration 1, thus resulting in a degenerate basic solution. The optimum is reached after an additional iteration is carried out.

What is the practical implication of degeneracy? Look at Figure 3-4, which provides the graphical solution to the model. Three lines pass through the optimum $x_1 = 0$, $x_2 = 2$. Because this is a two-dimensional problem, the point is *overdetermined* and one of the constraints is redundant. In practice, the mere knowledge that some resources are superfluous can prove valuable during the implementation of the solution. The in-

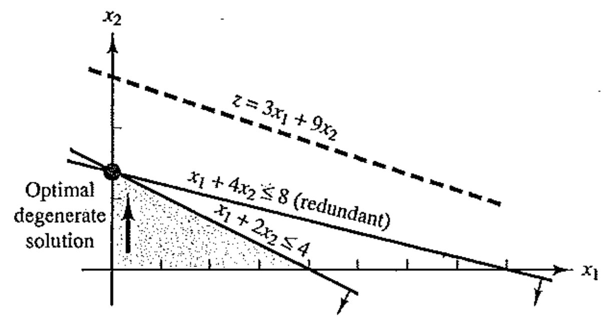

Figure 3-4

formation may also lead to discovering irregularities in the construction of the model. Unfortunately, there are no reliable techniques for identifying redundant constraints directly from the tableau.

From the theoretical standpoint, degeneracy has two implications. The first deals with the phenomenon of **cycling** or **circling**. If you look at iterations 1 and 2 in the tableaux, you will find that the objective value has not improved ($z = 18$). It is thus conceivable that the simplex procedure would repeat the *same sequence* of iterations, never improving the objective value and never terminating the computations. Although there are methods for eliminating cycling, these methods could lead to a drastic slowdown in computations. For this reason, most LP codes do not include provisions for cycling, relying on the fact that the percentage of such problems is too small to warrant a routine implementation of the cycling procedures.

The second theoretical point arises in the examination of iterations 1 and 2. Both iterations, although differing in classifying the variables as basic and nonbasic, yield identical values of all the variables and objective value, namely,

$$x_1 = 0,\ x_2 = 2,\ x_3 = 0,\ x_4 = 0,\ z = 18$$

Is it possible then to stop the computations at iteration 1 (when degeneracy first appears), even though it is not optimum? The answer is no, because the solution may be *temporarily* degenerate (see Problem 3.5a–2).

Problem set 3.5a

1. Consider the graphical solution space in Figure 3–5. Suppose that the simplex iterations start at A and that the optimum solution occurs at D and that the objective function is defined such that at A, x_1 enters the solution first.
   (a) Identify (on the graph) the extreme points that define the simplex method path to the optimum point.
   (b) Determine the maximum possible number of simplex iterations needed to reach the optimum solution.
2. Show (both graphically and by the simplex method) that the following LP is *temporarily* degenerate.

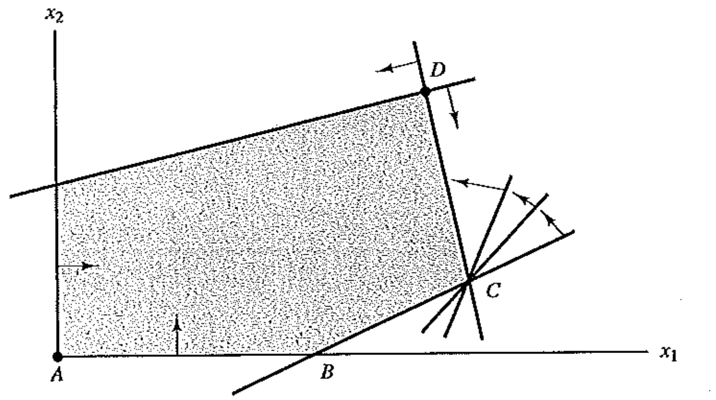

Figure 3–5

$$\text{Maximize } z = 3x_1 + 2x_2$$

subject to

$$4x_1 + 3x_2 \le 12$$
$$4x_1 + x_2 \le 8$$
$$4x_1 - x_2 \le 8$$
$$x_1, x_2 \ge 0$$

3. Use TORA's interactive option to "thumb" through the successive simplex iteration of the following LP (developed by E. M. Beale). The starting all-slack basic feasible solution will reappear identically in iteration 7. The example illustrates the occurrence of *cycling* in the simplex iterations and the possibility of the simplex algorithm never converging to the optimum solution.

$$\text{Maximize } z = \frac{3}{4}x_1 - 20x_2 + \frac{1}{2}x_3 - 6x_4$$

subject to

$$\frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \le 0$$
$$\frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 \le 0$$
$$x_3 \le 1$$
$$x_1, x_2, x_3, x_4 \ge 0$$

It is interesting that if all the coefficients in this LP are converted to integer values (by using proper multiples), then the simplex algorithm will reach the optimum in a finite number of iterations (try it!).

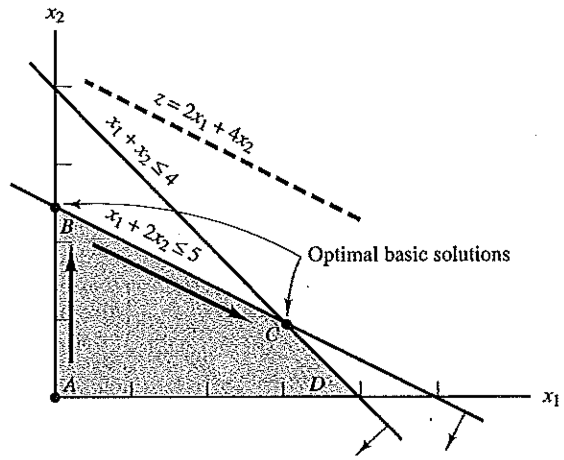

Figure 3–6

(Warning: Do not use TORA's automated option; otherwise, as expected, the iterations will cycle indefinitely.)

3.5.2 Alternative Optima

When the objective function is parallel to a *binding* constraint (i.e., a constraint that is satisfied as an equation by the optimal solution), the objective function will assume the *same optimal value* at more than one solution point. For this reason they are called **alternative optima.** The next example shows that there is an *infinity* of such solutions. The example also demonstrates the practical significance of encountering alternative optima.

Example 3.5-2 (INFINITY OF SOLUTIONS).

$$\text{Maximize } z = 2x_1 + 4x_2$$

subject to

$$x_1 + 2x_2 \le 5$$
$$x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

Figure 3-6 demonstrates how alternative optima can arise in the LP model when the objective function is parallel to a binding constraint. Any point on the *line segment BC* represents an alternative optimum with the same objective value $z = 10$.

The iterations of the model are given by the following tableaux.

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	−2	−4	0	0	0
x_2 enters	x_3	1	2	1	0	5
x_3 leaves	x_4	1	1	0	1	4
1 (optimum)	z	0	0	2	0	10
x_1 enters	x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$
2	z	0	0	2	0	10
(alternative	x_2	0	1	1	−1	1
optimum)	x_1	1	0	−1	2	3

Iteration 1 gives the optimum $x_1 = 0, x_2 = \frac{5}{2}$, and $z = 10$, which coincides with point B in Figure 3-6. How do we know from this tableau that the alternative optima exist? Look at the coefficients of the *non*basic variables in the z-equation of iteration 1. The coefficient of nonbasic x_1 is zero, indicating that x_1 can enter the basic solution without changing the value of z, but causing a change in the values of the variables. Iteration 2 does just that—letting x_1 enter the basic solution, which will force x_4 to leave. This results in the new solution point at $C(x_1 = 3, x_2 = 1, z = 10)$.

The simplex method determines only the two corner points B and C. Mathematically, we can determine all the points $(\hat{x}_1, \hat{x}_2)$ on the line segment BC as a nonnegative weighted average of the points B and C. Thus, given $0 \le \alpha \le 1$ and

$$B:\ x_1 = 0,\ \ x_2 = \frac{5}{2}$$

$$C:\ x_1 = 3,\ \ x_2 = 1$$

then all the points on the line segment BC are given by

$$\hat{x}_1 = \alpha(0) + (1 + \alpha)(3) = 3 - 3\alpha$$

$$\hat{x}_2 = \alpha\left(\frac{5}{2}\right) + (1 - \alpha)(1) = 1 + \frac{3}{2}\alpha$$

When $\alpha = 0$, $(\hat{x}_1, \hat{x}_2) = (3, 1)$, which is point C. When $\alpha = 1$, $(\hat{x}_1, \hat{x}_2) = (0, \frac{5}{2})$, which is point B. For values of α between 0 and 1, $(\hat{x}_1, \hat{x}_2)$ lies between B and C.

In practice, alternative optima are useful because they allow us to choose from many solutions without experiencing any deterioration in the objective value. In the example, for instance, the solution at B shows that activity 2 only is at a positive level, whereas at C both activities are positive. If the example represents a product-mix situation, it may be advantageous from the standpoint of sales competition to produce two products rather than one. In this case, the solution at C is recommended.

Problem set 3.5b

1. For the following LP, find three alternative optimal basic solutions, and then write a general expression for all the nonbasic alternative optima constituting these three basic solutions.

$$\text{Maximize } z = x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\le 10 \\ x_1 + x_2 &\le 5 \\ x_1 &\le 1 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

2. Show that all the alternative optima of the following LP are all nonbasic. Give a two-dimensional graphical demonstration of the type of solution space and objective function that will produce this result.

$$\text{Maximize } z = 2x_1 - x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 - x_2 + 5x_3 &\le 10 \\ 2x_1 - x_2 + 3x_3 &\le 40 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

3. For the following LP, show that the optimal solution is degenerate and that there exist alternative solutions that are all nonbasic.

$$\text{Maximize } z = 3x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\le 5 \\ x_1 + x_2 - x_3 &\le 2 \\ 7x_1 + 3x_2 - 5x_3 &\le 20 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

3.5.3 Unbounded Solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraints, meaning that the solution space is **unbounded** in at least one direction. As a result, the objective value may increase (maximization case) or decrease (minimization case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded.

Unboundedness in a model can point to one thing only: The model is poorly constructed. The most likely irregularities in such models are that one or more nonredundant constraints are not accounted for, and the parameters (constants) of some constraints are not estimated correctly.

The following examples show how unboundedness, both in the solution space and the objective value, can be recognized in the simplex tableau.

Example 3.5-3 (UNBOUNDED OBJECTIVE VALUE).

$$\text{Maximize } z = 2x_1 + x_2$$

subject to

$$x_1 - x_2 \le 10$$
$$2x_1 \le 40$$
$$x_1, x_2 \ge 0$$

Starting Iteration.

Basic	x_1	x_2	x_3	x_4	Solution
z	−2	−1	0	0	0
x_3	1	−1	1	0	10
x_4	2	0	0	1	40

In the starting tableau, both x_1 and x_2 are candidates for entering the solution. Because x_1 has the most negative coefficient, it is normally selected as the entering variable. However, *all* the *constraint* coefficients under x_2 are *negative* or *zero*, meaning that x_2 can be increased indefinitely without violating any of the constraints. Because each unit increase in x_2 will increase z by 1, an infinite increase in x_2 will also result in an infinite increase in z. Thus, the problem has no bounded solution. This result can be seen in Figure 3-7. The solution space is unbounded in the direction of x_2, and the value of z can be increased indefinitely.

The rule for recognizing unboundedness is as follows. If at any iteration the constraint coefficients of any *nonbasic* variable are nonpositive, then the *solution space* is unbounded in that direction. If, in addition, the objective coefficient of that variable is negative in the case of maximization or positive in the case of minimization, then the *objective value* also is unbounded.

Problem set 3.5c

1. In Example 3.5-3, show that if, according to the optimality condition, we start with x_1 as the entering variable, then the simplex algorithm will lead eventually to an unbounded solution.

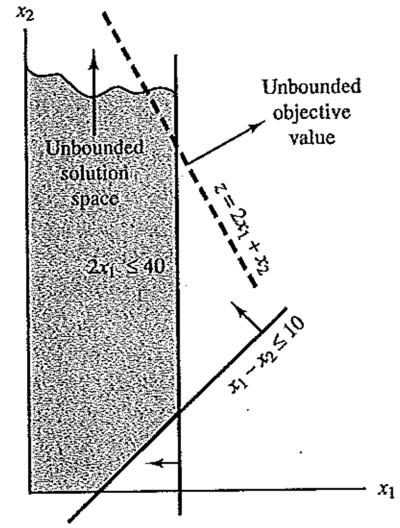

Figure 3-7

2. Consider the LP:

$$\text{Maximize } z = 20x_1 + 10x_2 + x_3$$

subject to

$$3x_1 - 3x_2 + 5x_3 \le 50$$
$$x_1 + x_3 \le 10$$
$$x_1 - x_2 + 4x_3 \le 20$$
$$x_1, x_2, x_3 \ge 0$$

(a) By inspecting the constraints, determine the direction (x_1, x_2, or x_3) in which the solution space is unbounded.
(b) Without further computations, what can you conclude regarding the optimum objective value?

3. In some ill-constructed LP models, the solution space may be unbounded even though the problem may have a bounded objective value. Such an occurrence can point only to irregularities in the construction of the model. In large problems, it may be difficult to detect the existence of unboundedness by inspection. Devise a procedure for determining whether or not a solution space is unbounded, and apply it to the following model:

$$\text{Maximize } z = 40x_1 + 20x_2 + 2x_3$$

subject to

$$3x_1 - 3x_2 + 5x_3 \le 50$$
$$x_1 \qquad + x_3 \le 10$$
$$x_1 - x_2 + 4x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

3.5.4 Infeasible Solution

If the constraints are not satisfied simultaneously, the model has no feasible solution. This situation can never occur if *all* the constraints are of the type $\le$ (assuming non-negative right-hand-side constants), because the slacks provide a *feasible* solution. For other types of constraints, we use artificial variables. Although the artificials are penalized to force them to zero at the optimum, this can occur only if the model has a feasible space. Otherwise, at least one artificial variable will be *positive* in the optimum iteration.

From the practical standpoint, an infeasible space points to the possibility that the model is not formulated correctly.

Example 3.5-4 (INFEASIBLE SOLUTION SPACE).

$$\text{Maximize } z = 3x_1 + 2x_2$$

subject to

$$2x_1 + x_2 \le 2$$
$$3x_1 + 4x_2 \ge 12$$
$$x_1, x_2 \ge 0$$

The following tableaux provides the simplex iterations of the model.

Iteration	Basic	x_1	x_2	x_4	x_3	R	Solution
0	z	$-3-3M$	$-2-4M$	M	0	0	$-12M$
x_2 enters	x_3	2	1	0	1	0	2
x_3 leaves	R	3	4	-1	0	1	12
1	z	$1+5M$	0	M	$2+4M$	0	$4-4M$
(pseudo-	x_2	2	1	0	1	0	2
optimum)	R	-5	0	-1	-4	1	4

Optimum iteration 1 shows that the artificial variable R is *positive* (= 4) which indicates that the problem is infeasible. Figure 3-8 demonstrates the infeasible solution space. The simplex method, by allowing the artificial variable to be positive, in essence has reversed the direction of the inequality from $3x_1 + 4x_2 \ge 12$ to $3x_1 + 4x_2 \le 12$. (Can you explain how?) The result is what we may call a **pseudo-optimal solution.**

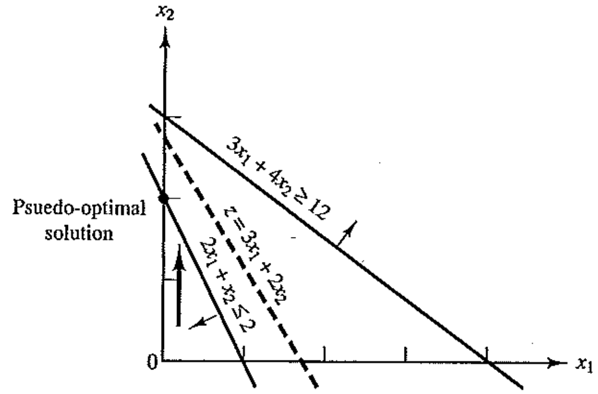

Figure 3-8

Problem set 3.5d

1. Toolco produces three types of tools, $T1$, $T2$, and $T3$. The tools use two raw materials, $M1$ and $M2$, according to the data in the following table:

	Number of units of raw materials per tool		
Raw material	$T1$	$T2$	$T3$
$M1$	3	5	6
$M2$	5	3	4

The daily availability of raw materials is 1000 units and 1200 units, respectively. The manager in charge of production was informed by the marketing department that according to their research, the daily demand for all three tools must be at least 500 units. Would the manufacturing department be able to satisfy the demand? If not, what is the most Toolco can provide of the three tools?

2. Consider the LP model

$$\text{Maximize } z = 3x_1 + 2x_2 + 3x_3$$
$$2x_1 + x_2 + x_3 \le 2$$
$$3x_1 + 4x_2 + 2x_3 \ge 8$$
$$x_1, x_2, x_3 \ge 0$$

Show by the M-technique that the optimal solution includes an artificial basic variable. However, because its value is zero, the problem has a *feasible* optimal solution.