of one decision cannot be assessed until we know how others are resolved. Such circumstances often lead to quadratic assignment models.

11.16 **Quadratic assignment models** minimize or maximize a quadratic objective function of the form

$$\sum_{i}\sum_{j}\sum_{k>i}\sum_{\ell\neq j} c_{i,j,k,\ell}\, x_{i,j}\, x_{k,\ell}$$

subject to assignment constraints $\boxed{11.14}$, where $c_{i,j,k,\ell}$ is the cost (or benefit) of assigning i to j and k to ℓ.

Notice that each objective function term

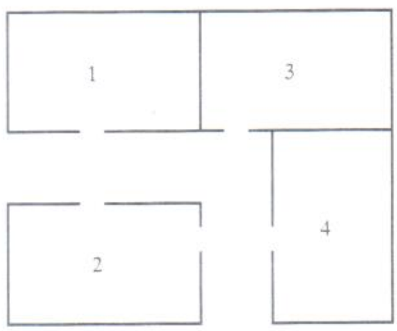
$$c_{i,j,k,\ell} \cdot x_{i,j} \cdot x_{k,\ell}$$

involves two assignment decisions. Cost $c_{i,j,k,\ell}$ is realized only if both $x_{i,j} = 1$ and $x_{k,\ell} = 1$. That is, $c_{i,j,k,\ell}$ applies only if i is assigned to j and k is assigned to ℓ.

Example 11.5: Mall Layout Quadratic Assignment

Some of the most common cases producing quadratic assignment models arise in **facility layout**. We are given a collection of machines, offices, departments, stores, and so on, to arrange within a facility, and a set of locations within which they must fit. The problem is to decide which unit to assign to each location.

Figure 11.3 illustrates with 4 possible locations for stores in a shopping mall. Walking distances (in feet) between the shop locations are displayed in the adjacent table. The 4 prospective tenants for the shop locations are listed in Table 11.5. The table also shows the number of customers each week (in thousands) who might wish to visit various pairs of shops. For example, a projected 5 thousand customers per week will visit both 1 (Clothes Are) and 2 (Computers Aye).



Distance (feet)				
$j \backslash \ell$	1	2	3	4
1	—	80	150	170
2	80	—	130	100
3	150	130	—	120
4	150	100	120	—

Figure 11.3 Mall Layout Example Locations

Mall managers want to arrange the stores in the 4 locations to minimize customer inconvenience. One very common measure is **flow-distance**, the product of flow volumes between facilities and the distances between their assigned locations. For example, if shop 1 (Clothes Are) is located in space 1, and shop 4 (Book Bazaar) is located in space 2, their 7 thousand common customers will have to walk the 80 feet between the locations. This adds $7 \cdot 80 = 560$ thousand customer-feet to the flow-distance.

TABLE 11.5 Mall Layout Example Tenants

	Common Customers with k (000's)			
Store, i	**1**	**2**	**3**	**4**
1: Clothes Are	—	5	2	7
2: Computers Aye	5	—	3	8
3: Toy Parade	2	3	—	3
4: Book Bazaar	7	8	3	—

Mall Layout Example Model

Notice that the flow-distance for any pair of shops cannot be computed until we know where both are assigned. This is the assignment combinations characteristic that yields quadratic assignment models.

Using the decision variables

$$x_{i,j} \triangleq \begin{cases} 1 & \text{if shop } i \text{ is assigned to location } j \\ 0 & \text{otherwise} \end{cases}$$

the required quadratic assignment model is

$$\begin{aligned}
\min \quad & 5(80x_{1,1}x_{2,2} + 150x_{1,1}x_{2,3} + 170x_{1,1}x_{2,4} && \text{(shops 1 and 2)} \\
& \quad + 80x_{1,2}x_{2,1} + 130x_{1,2}x_{2,3} + 100x_{1,2}x_{2,4} \\
& \quad + 150x_{1,3}x_{2,1} + 130x_{1,3}x_{2,2} + 120x_{1,3}x_{2,4} \\
& \quad + 170x_{1,4}x_{2,1} + 100x_{1,4}x_{2,2} + 120x_{1,4}x_{2,3}) \\
& 2(80x_{1,1}x_{3,2} + 150x_{1,1}x_{3,3} + 170x_{1,1}x_{3,4} && \text{(shops 1 and 3)} \\
& \quad + 80x_{1,2}x_{3,1} + 130x_{1,2}x_{3,3} + 100x_{1,2}x_{3,4} \\
& \quad + 150x_{1,3}x_{3,1} + 130x_{1,3}x_{3,2} + 120x_{1,3}x_{3,4} \\
& \quad + 170x_{1,4}x_{3,1} + 100x_{1,4}x_{3,2} + 120x_{1,4}x_{3,3}) \\
& 7(80x_{1,1}x_{4,2} + 150x_{1,1}x_{4,3} + 170x_{1,1}x_{4,4} && \text{(shops 1 and 4)} \\
& \quad + 80x_{1,2}x_{4,1} + 130x_{1,2}x_{4,3} + 100x_{1,2}x_{4,4} && \text{(11.1} \\
& \quad + 150x_{1,3}x_{4,1} + 130x_{1,3}x_{4,2} + 120x_{1,3}x_{4,4} \\
& \quad + 170x_{1,4}x_{4,1} + 100x_{1,4}x_{4,2} + 120x_{1,4}x_{4,3}) \\
& 3(80x_{2,1}x_{3,2} + 150x_{2,1}x_{3,3} + 170x_{2,1}x_{3,4} && \text{(shops 2 and 3)} \\
& \quad + 80x_{2,2}x_{3,1} + 130x_{2,2}x_{3,3} + 100x_{2,2}x_{3,4} \\
& \quad + 150x_{2,3}x_{3,1} + 130x_{2,3}x_{3,2} + 120x_{2,3}x_{3,4} \\
& \quad + 170x_{2,4}x_{3,1} + 100x_{2,4}x_{3,2} + 120x_{2,4}x_{3,3}) \\
& 8(80x_{2,1}x_{4,2} + 150x_{2,1}x_{4,3} + 170x_{2,1}x_{4,4} && \text{(shops 2 and 4)} \\
& \quad + 80x_{2,2}x_{4,1} + 130x_{2,2}x_{4,3} + 100x_{2,2}x_{4,4} \\
& \quad + 150x_{2,3}x_{4,1} + 130x_{2,3}x_{4,2} + 120x_{2,3}x_{4,4} \\
& \quad + 170x_{2,4}x_{4,1} + 100x_{2,4}x_{4,2} + 120x_{2,4}x_{4,3})
\end{aligned}$$

$$\begin{aligned}
&3(80x_{3,1}x_{4,2} + 150x_{3,1}x_{4,3} + 170x_{3,1}x_{4,4} && \text{(shops 3 and 4)}\\
&\quad + 80x_{3,2}x_{4,1} + 130x_{3,2}x_{4,3} + 100x_{3,2}x_{4,4}\\
&\quad + 150x_{3,3}x_{4,1} + 130x_{3,3}x_{4,2} + 120x_{3,3}x_{4,4}\\
&\quad + 170x_{3,4}x_{4,1} + 100x_{3,4}x_{4,2} + 120x_{3,4}x_{4,3})\\
\text{s.t.}\quad & x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 1 && \text{(1, Clothes Are)}\\
& x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 1 && \text{(2, Computers Aye)}\\
& x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} = 1 && \text{(3, Toy Parade)}\\
& x_{4,1} + x_{4,2} + x_{4,3} + x_{4,4} = 1 && \text{(4, Book Bazaar)}\\
& x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1} = 1 && \text{(location 1)}\\
& x_{1,2} + x_{2,2} + x_{3,2} + x_{4,2} = 1 && \text{(location 2)}\\
& x_{1,3} + x_{2,3} + x_{3,3} + x_{4,3} = 1 && \text{(location 3)}\\
& x_{1,4} + x_{2,4} + x_{3,4} + x_{4,4} = 1 && \text{(location 4)}\\
& x_{i,j} = 0 \text{ or } 1 \quad i = 1, \ldots, 4; j = 1, \ldots, 4
\end{aligned}$$

The objective function computes total flow distance for all pairs of shops and all possible assigned locations. Assignment constraints assure that one shop goes to each location and each locations gets one shop. An optimal assignment places shop 1 in location 1, shop 2 in location 4, shop 3 in location 3, and shop 4 in location 2, for a total flow distance of 3260 thousand customer-feet.

SAMPLE EXERCISE 11.11: FORMULATING QUADRATIC ASSIGNMENT MODELS

An industrial engineer has divided a proposed machine shop's floor area into 12 grid squares, g, each of which will be the location of a single machine m. He has also estimated the distance, $d_{g,g'}$, between all pairs of grid squares and the number of units, $f_{m,m'}$, that will have to travel between machines m and m' (in both directions) during each week of operation. Formulate a quadratic assignment model to layout the shop in a way that will minimize material handling cost (i.e., minimize the product of between machine flows and the distance between their locations). Assume $d_{g,g'} = d_{g',g}$.

Modeling: Using the decision variables

$$x_{m,g} \triangleq \begin{cases} 1 & \text{if machine } m \text{ is located at grid square } g \\ 0 & \text{otherwise} \end{cases}$$

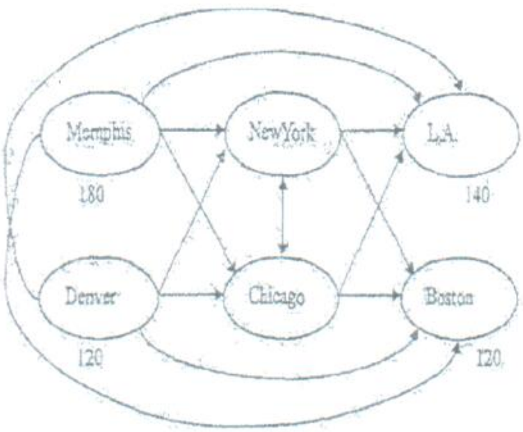
the required model is

$$\begin{aligned}
\min\quad & \sum_{m=1}^{12}\sum_{g=1}^{12}\sum_{m'>m}^{12}\sum_{\substack{g'=1\\ g'\neq g}}^{12} f_{m,m'}d_{g,g'}x_{m,g}x_{m',g'} && \text{(flow distance)}\\
\text{s.t.}\quad & \sum_{g=1}^{12} x_{m,g} = 1 \qquad m = 1, \ldots, 12 && \text{(square per machine)}\\
& \sum_{m=1}^{12} x_{m,g} = 1 \qquad g = 1, \ldots, 12 && \text{(machine per square)}\\
& x_{i,j} = 0 \text{ or } 1 \qquad m = 1, \ldots, 12; \quad g = 1, \ldots, 12
\end{aligned}$$

§轉運問題§ THE TRANSSHIPMENT MODEL

標準運輸模式假設，供給點與運送點之間的路徑為最短路徑，而轉運問題結合運輸模式與最短路徑成為一方法。

轉運問題乃基於運輸上及管理上的需求而產生。例如某地区定期航線的港口之間，經常運用轉運方式，以湊足開船的載量，經常有甲港口輸往非洲地区之貨物集中到乙港口以便集中一船運送。



	Memphis	Denver	N.Y.	Chicago	L.A.	Boston
Memphis	$0	—	$8	$13	$25	$28
Denver	—	$0	$15	$12	$26	$25
N.Y.	—	—	$0	$6	$16	$17
Chicago	—	—	$6	$0	$14	$16
L.A.	—	—	—	—	$0	—
Boston	—	—	—	—	—	$0

Memphis 與 Denver 為供給地，L.A.與 Boston 為需求地，其相關供給量、需求量、轉運點與轉運成本如上所述，試解決上述問題，列出轉運問題模式，無需求解

	New York	Chicago	Boston	L.A.	
Memphis	8	13	25	28	180
Denver	15	12	26	25	120
New York	0	6	16	17	300
Chicago	6	0	14	16	300
	300	300	120	140	

We now describe how the optimal solution to a transshipment problem can be found by solving a transportation problem. Given a transshipment problem, we create a balanced transportation problem by the following procedure (assume that total supply exceeds total demand):

Step 1 If necessary, add a dummy demand point (with a supply of 0 and a demand equal to the problem's excess supply) to balance the problem. Shipments to the dummy and from a point to itself will, of course, have a zero shipping cost. Let s = total available supply.

Step 2 Construct a transportation tableau as follows: A row in the tableau will be needed for each supply point and transshipment point, and a column will be needed for each demand point and transshipment point. Each supply point will have a supply equal to its original supply, and each demand point will have a demand equal to its original demand. Let s = total available supply. Then each transshipment point will have a supply equal to (point's original supply) + s and a demand equal to (point's original demand) + s. This ensures that any transshipment point that is a net supplier will have a net outflow equal to the point's original supply, and, similarly, a net demander will have a net inflow equal to the point's original demand. Although we don't know how much will be shipped through each transshipment point, we can be sure that the total amount will not exceed s. This explains why we add s to the supply and demand at each transshipment point. By adding the same amounts to the supply and demand, we ensure that the net outflow at each transshipment point will be correct, and we also maintain a balanced transportation tableau.

Transshipment occurs in the network in Figure 5.7 because the entire supply amount of 2200 (= 1000 + 1200) cars at nodes $P1$ and $P2$ could conceivably pass through any node of the network before ultimately reaching their destinations at nodes $D1$, $D2$, and $D3$. In this regard, each node of the network with both input and output arcs ($T1$, $T2$, $D1$, and $D2$) acts as both a source and a destination and is referred to as a **transshipment node**. The remaining nodes are either **pure supply nodes** ($P1$ and $P2$) or **pure demand nodes** ($D3$).

The transshipment model can be converted into a regular transportation model with six sources ($P1$, $P2$, $T1$, $T2$, $D1$, and $D2$) and five destinations ($T1$, $T2$, $D1$, $D2$, and $D3$). The amounts of supply and demand at the different nodes are computed as

$$\begin{aligned}
\text{Supply at a } \textit{pure supply node} &= \text{Original supply}\\
\text{Demand at a } \textit{pure demand node} &= \text{Original demand}\\
\text{Supply at a } \textit{transshipment node} &= \text{Original supply} + \text{Buffer amount}\\
\text{Demand at a } \textit{transshipment node} &= \text{Original demand} + \text{Buffer amount}
\end{aligned}$$

The buffer amount should be sufficiently large to allow all of the *original* supply (or demand) units to pass through any of the *transshipment* nodes. Let B be the desired buffer amount; then

$$\begin{aligned}
B &= \text{Total supply (or demand)}\\
&= 1000 + 1200 \text{ (or } 800 + 900 + 500)\\
&= 2200 \text{ cars}
\end{aligned}$$

Using the buffer B and the unit shipping costs given in the network, we construct the equivalent regular transportation model as in Table 5.43.

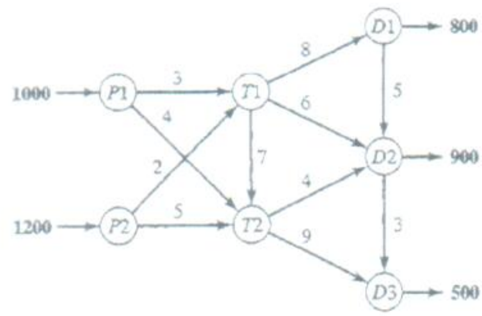


FIGURE 5.7
Transshipment network between plants and dealers

TABLE 5.43 Transshipment Model

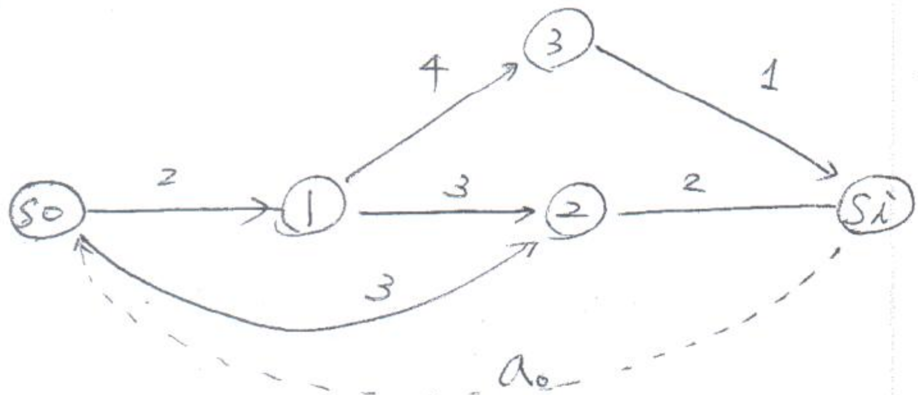
	$T1$	$T2$	$D1$	$D2$	$D3$	
$P1$	3	4	M	M	M	1000
$P2$	2	5	M	M	M	1200
$T1$	0	7	8	6	M	B
$T2$	M	0	M	4	9	B
$D1$	M	M	0	5	M	B
$D2$	M	M	M	0	3	B
	B	B	$800 + B$	$900 + B$	500	

The solution of the resulting transportation model (determined by TORA) is shown in Figure 5.8. Note the effect of transshipment: Dealer $D2$ receives 1400 cars, keeps 900 cars to sat-

Maximum Flow Problems 最大流量(川流問題)

在許多情況下，網路之節點可視為有容量限制去建構模式。在這些情形下，通常想從一 starting point 起点(source 源点)運送最大之流量至一 terminal point 終点(sink 匯点)。這類之問題，諸如，自來水同一時間之最大流量，高速公路之路網，最大流量等。

e.g.



Sunco Oil 公司想透过 pipeline 從 so 至 si 運送最大量的石油(在每小時)網路上之數字代表每小時可運过該節點之最大流量(每小時百万 barrels)由於管道的口徑大小不同，最大流量問題

就是要決定，每小時從 S_0 至 S_i 最大流量。

令 X_{ij} 代表每小時會通过弧線 (i,j)

百万 barrels.

X_0：flow 通过 artificial arc, i.e.

進入 sink 的流量 a_0

$$\text{Max} \quad Z = X_0$$

s.t.

$$X_{so,1} \le 2$$

$$X_{so,2} \le 3$$

$$X_{1,2} \le 3$$

$$X_{2,si} \le 2$$

$$X_{1,3} \le 4$$

$$X_{3,si} \le 1$$

(Arc capacity constraints)

$$X_0 = X_{so,1} + X_{so,2}$$ Node so flow constraint

$$X_{so,1} = X_{12} + X_{13}$$ Node 1 flow constraint

$$X_{so,2} + X_{12} = X_{2,si}$$ Node 2 flow 〃

$$X_{13} = X_{3,si}$$ Node 3 flow 〃

$$X_{3,si} + X_{2,si} = X_0$$ Node si flow 〃

conservation-of-flow constraint

$$X_{ij} \ge 0$$

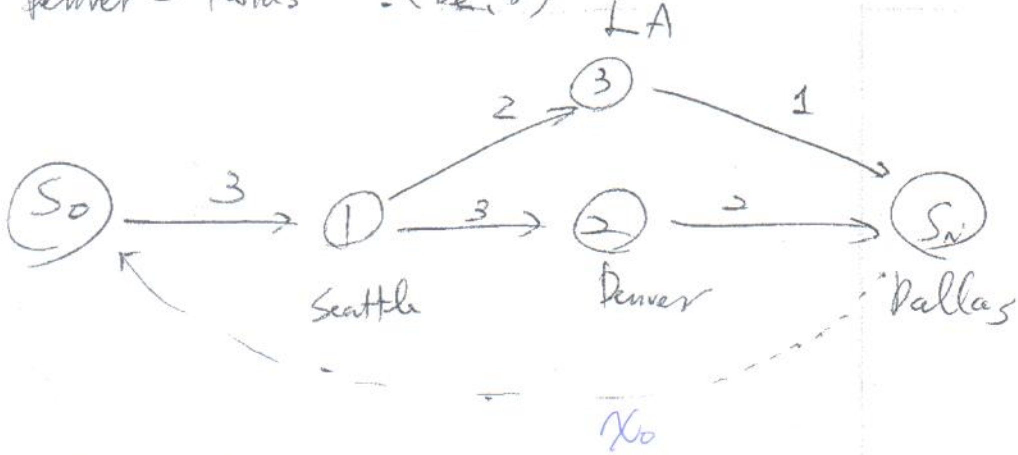
Optimal $Z=3$, $X_{so,1}=2$, $X_{so,2}=1$, $X_0=3$

$X_{1,3}=1$ $X_{3,si}=1$

$X_{1,2}=1$, $x_{2,si}=2$

e.g.

Cities	Maximum Number of Daily Flights
Juneau - Seattle (J, S)	3
Seattle - L.A. (S, L)	2
Seattle - Denver (S, De)	3
LA - Dallas (L, D)	1
Denver - Dallas (De, D)	2



$Z = X_0 = 3 \quad X_{JS} = 3 \quad X_{SL} = 1 \quad X_{SDE} = 2 \quad X_{LD} = 1 \quad X_{DE,D} = 2$

Fly-by-Night Airlines 必需決定 how many connecting flights can be arranged between Juneau, Alaska and Dallas, Texas, 由於 Landing space 有限, 每日可降落之最大班次如表顯示. Setup a maximum flow problem to maximize the number of connecting flights daily from Juneau to Dallas.

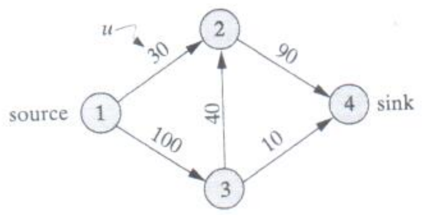
Max $Z = X_0$

s.t.

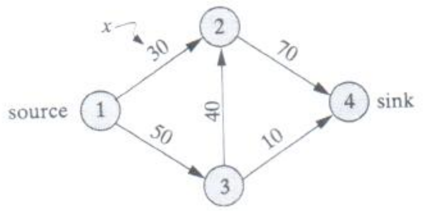
$X_{so,1} \le 3$	$X_0 = X_{so,1}$
$X_{1,3} \le 2$	$X_{so,1} = X_{1,3} + X_{1,2}$
$X_{1,2} \le 3$	$X_{1,2} = X_{2,si}$
$X_{3,si} \le 1$	$X_{13} = X_{3,si}$
$X_{2,si} \le 2$	$X_{3,si} + X_{2,si} = X_0$

SAMPLE EXERCISE 10.17: IDENTIFYING MAXIMUM FLOWS

Determine by inspection the maximum flow from node 1 to node 4 in the following graph (numbers on arcs are capacities $u_{i,j}$):



Analysis: Careful examination of the possibilities will establish that a maximum flow sends 80 units from 1 to 4 as follows:



EXAMPLE 10.4: BUILDING EVACUATION MAXIMUM FLOW

Maximum flow problems arise most often as subproblems in more complex operations research studies. However, they occur naturally in evaluating the safety of proposed building designs.[3] Proper design requires adequate capacity for building evacuation in the event of an emergency.

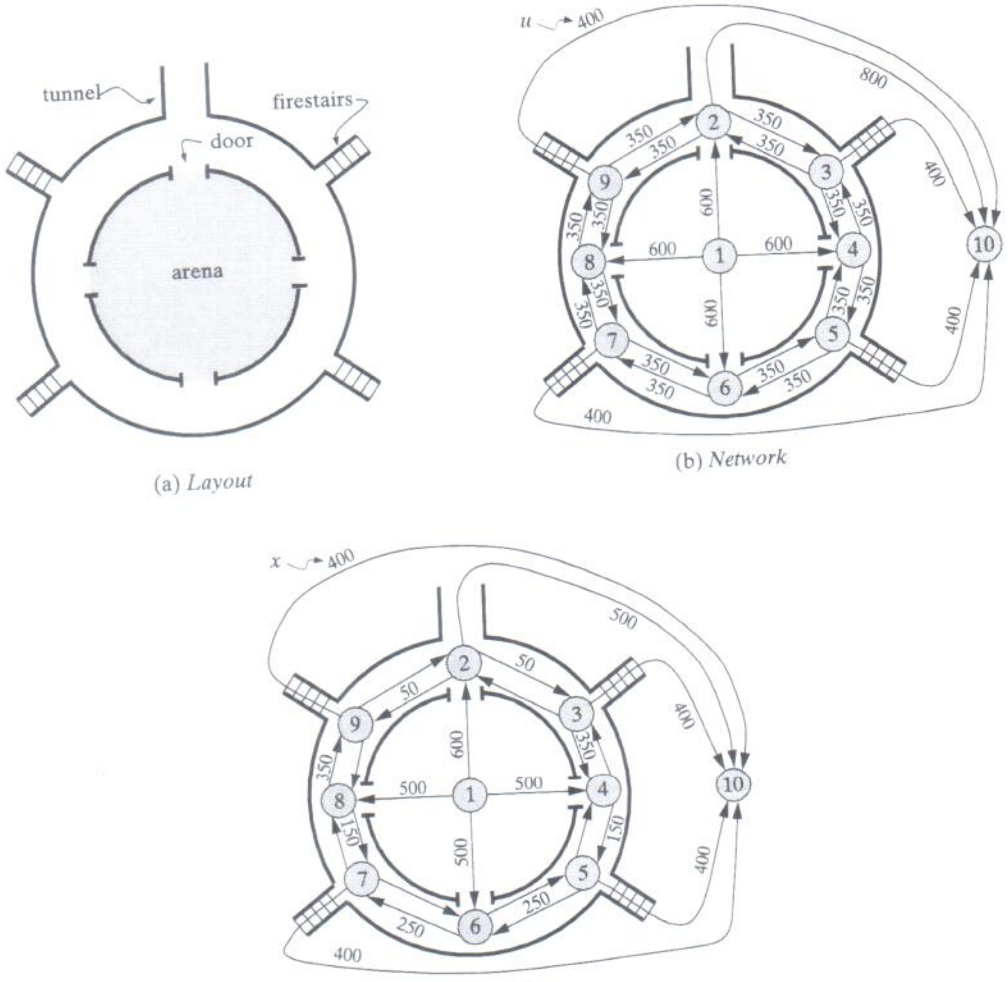
Figure 10.14 shows a small example involving a proposed sports arena. Patrons in the arena would exit in an emergency through doors on all four sides that can accommodate 600 persons per minute. Those doors lead into an outer hallway that can move 350 persons per minute in each direction. Egress from the hallway is through four firestairs with capacity 400 persons per minute and a tunnel to the parking lot accommodating 800 persons per minute. Our interest is in the maximum rate of evacuation possible with this design.

Part (b) of Figure 10.14 shows how we reduce this safety analysis to a maximum flow model. Patron flows originate at source node 1. Outbound arcs model the four doorways. The flows around the outer hall lead to the four stairways and the tunnel. Persons exiting by any of those means pass to sink node 10. Capacities enforce the flow rates of the various facilities.

We wish to know the maximum flow from 1 to 10, subject to the capacities indicated. An optimal flow is provided in the arc labels of part (b). Patrons can escape at a total rate of 2100 per minute.

---

[3] Based in part on L. G. Chalmet, R. L. Frances, and P. B. Saunders (1982), "Network Models for Building Evacuation," *Management Science*, 28, 86–105. All numerical data and diagrams were made up by the author of this book.



(a) *Layout*

(b) *Network*

(c) *Optimal flow*

FIGURE 10.14 Building Evacuation Maximum Flow Example

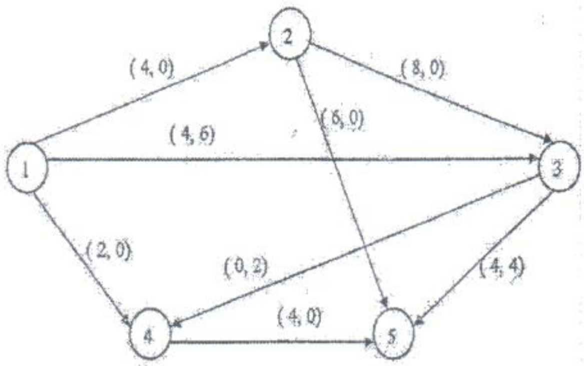
Return Arc Network Flow Formulation of Maximum Flow Problems

As so far presented, neither the tiny example of Sample Exercise 10.17 nor the larger one of Figure 10.14(b) is a minimum cost network flow problem. Flow conservation and capacity requirements are much like standard model $\boxed{10.3}$, but we have specified no costs, and flows do not balance at source and sink.

To create a true minimum cost flow problem, we add a return arc.

$\boxed{10.31}$ **Return arcs** balance unknown source-to-sink flows by feeding back that flow in an artificial arc from sink to source.

Adding a return arc to the maximum flow example of Figure 10.14(b) produces the following digraph:



最大流量問題可用下述演算法求解：

步驟1　找出一條由源點至滙點可容納一正流量物品的路徑。若不存在，跳至步驟5 。

步驟2　算出沿這條路徑可裝運的最大容量，並以 k 表示之。

步驟3　這條路徑中，每一分枝的順向容量（即順著 k 單位流量的方向之容量）均減少 k 單位，而逆向容量均增加 k 。滙點的運送量也增加 k 單位。

步驟4　回到步驟1 。

步驟5　最大流量即為滙點的運送量。比較原網路與終期網路，將任一容量的減少視為裝運，則可決定最優裝運計畫。

Minimum Cost Network Flow Problems 最小成本網路流量問題

Transportation, assignment, transshipment, shortest path, maximum flows (最大流量) and CPM are all special cases of the minimum cost network flow problem (MCNFP). 任何的 MCNFP 可以用 Network Simplex 求解 (網路单体法)。

MCNFP

$$\min \sum_{\text{all arcs}} C_{ij} X_{ij}$$

$$\text{s.t.} \quad \sum_{j} X_{ij} - \sum_{k} X_{ki} = b_i \quad (\text{for each node } i \text{ in the network})$$

$$L_{ij} \le X_{ij} \le U_{ij} \quad (\text{for each arc in the network})$$

其中 X_{ij}: 從 node i 至 node j 透过 arc (i,j) 之流量 (number of units of flow)

b_i: node i 之淨供給 (outflow − inflow)

C_{ij}: 從 node i 至 node j 透过 arc (i,j) 運送每一单位流量之運送成本

L_{ij} = arc (i, j) 流量下限，如果沒有 Lower bound)

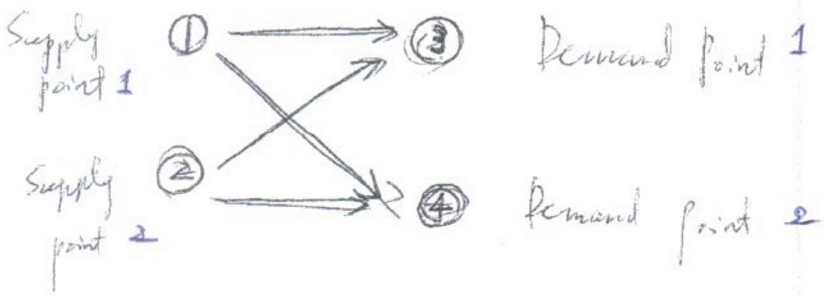
$L_{ij} = \infty$.

U_{ij} = arc (i, j) 流量上限，如果沒有上限 Upper bound，$U_{ij} = \infty$.

· 第一條限制式，限制節點 i 之淨流量 (流出 − 流入) 必需等於 b_i ⇒ 這條限制式稱為 flow balance equations (網路流量均衡限制式)

Formulating a Transportation Problem as an MCNFP.

1	2	4	(node 1)	$b_1 = 4$	4 − 0
3	4	5	(node 2)	$b_2 = 5$	5 − 0
6	3				
(Node 3)	(Node 4)				
$b_3 = -6$	$b_4 = -3$				
0 − 6	0 − 3				

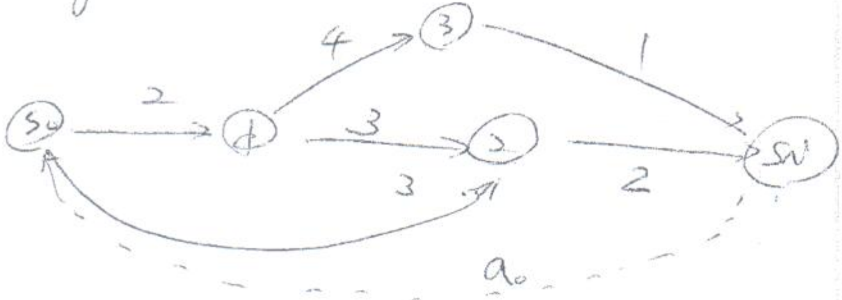


$$\text{Min } z = x_{13} + 2x_{14} + 3x_{23} + 4x_{24}$$

$x_{13} + x_{14} = 4$	node 1	流出-流入
$x_{23} + x_{24} = 5$	node 2	
$-x_{13} - x_{23} = -6$	node 3	$\boxed{\text{流出-流入 淨供給}}$
$-x_{14} - x_{24} = -3$	node 4	

All variables nonnegatives

Formulating a Maximum Flow Problem as an MCNFP.



在加入 a_0 後, $b_{so} = b_1 = b_2 = b_3 = b_{si} = 0$.

重要性質

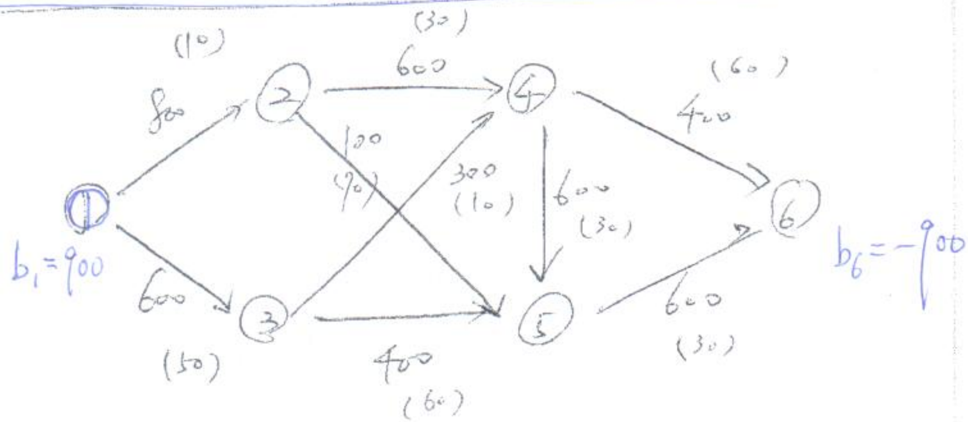
$$x_{ij}\text{'s} \begin{cases} +1 & \text{in the node } i \text{ flow balance equation, 供給} \\ -1 & \text{in the node } j \text{ flow balance equation, 需求} \\ 0 & \text{in all other flow balance equations.} \end{cases}$$

$$\text{Min } z = -X_0$$

$x_{so,1} + x_{so,2} - x_0 = 0$	node so
$x_{13} + x_{12} - x_{so,1} = 0$	node 1
$-x_{so,2} - x_{1,2} + x_{2,si} = 0$	node 2
$-x_{1,3} + x_{3,si} = 0$	node 3
$x_0 - x_{3,si} - x_{2,si} = 0$	node si

Arc (S. 1)

$$x_{so,1} \le 2$$
$$x_{so,2} \le 3$$
$$x_{13} \le 4$$
$$x_{12} \le 3$$
$$x_{3,si} \le 1$$
$$x_{2,si} \le 2$$



每小時平均有900 cars 進入 Network at node 1, and seek to travel to node 6. ( )中數字代表旅行時間。正的數字代表可通行之最大數目車流量 (within one hour). Formulate MCNFP that minimizes the total time required for all cars to travel from node 1 to node 6.

令 X_{ij}：每小時從 node i 至 node j 車輛數目

$$\text{Min } z = 10x_{12} + 50x_{13} + 70x_{25} + 30x_{24} + 30x_{56} + 30x_{45}$$

$$+60x_{46} + 60x_{45} + 10x_{34},$$

$b_1 = 900 \quad b_2 = b_3 = b_4 = b_5 = 0 \quad b_6 = -900$ (we will not introduce the artifical arc connecting node 6 to node 1).

node 1 $\quad x_{12} + x_{13} = 900$

node 2 $\quad x_{24} + x_{25} - x_{12} = 0$

node 3 $\quad x_{34} + x_{35} - x_{13} = 0$

node 4 $\quad x_{46} + x_{45} - x_{24} - x_{34} = 0$

node 5 $\quad x_{56} - x_{45} - x_{25} - x_{35} = 0$

node 6 $\quad -x_{46} - x_{56} = -900$

$x_{12} \le 800$

$x_{13} \le 600$

$x_{24} \le 600$

$x_{45} \le 100$

$x_{34} \le 300$

$x_{35} \le 400$

$x_{46} \le 400$

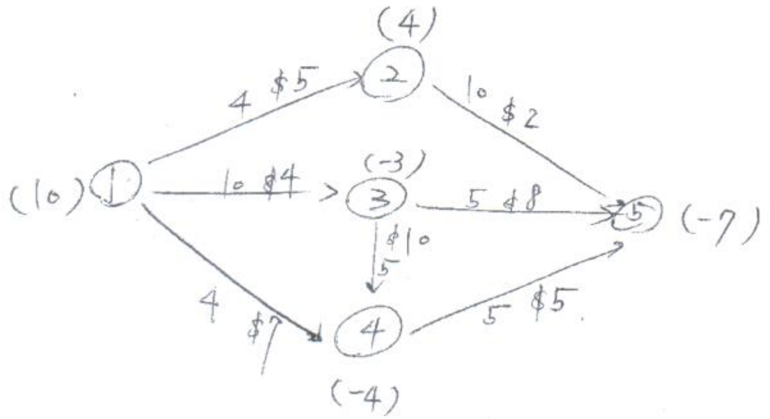
$x_{45} \le 600$

$x_{56} \le 600$

all x_{ij} nonnegative

Network Simplex Method 網路單体法

專門用來解決 minimum cost flow problem，如同 simplex method 在每一步驟找到 entering variable, leaving variable 解新的 Basic Feasible，只是透過一种特殊之結構 無需一矩陣表格型式 (simplex tableau)。



$C_{12}=5$ $b_1=10$
$C_{25}=2$ $b_2=4$
$C_{13}=4$ $b_3=-3$
$C_{35}=8$ $b_4=-4$
$C_{34}=10$ $b_5=-7$
$C_{45}=5$
$C_{14}=7$

$U_{12}=4$, $U_{25}=10$, $U_{13}=10$, $U_{35}=5$, $U_{14}=4$, $U_{34}=5$, $U_{45}=5$

為使用 Network Simplex 通常 $\sum b_i = 0$，可用 dummy node 確保達成。

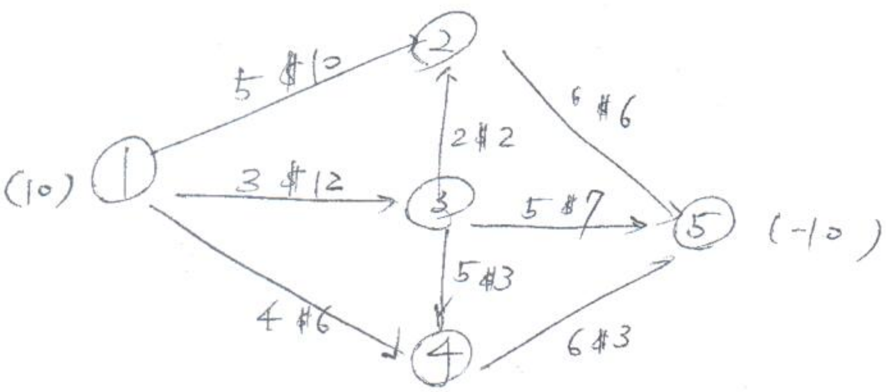
Basic Feasible Solution for MCNFPs will contain three types of variables:

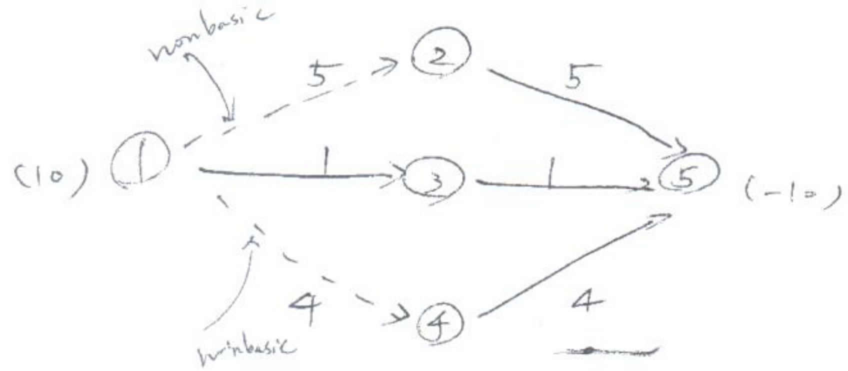
1. Basic variables: Without degeneracy, X_{ij} 滿足 $L_{ij} < X_{ij} < U_{ij}$, with degenerancy. X_{ij} 可能等於 arc(i, j)'s upper or lower bound.

2. Nonbasic variables X_{ij}: These equal arc(i, j)'s upper bound U_{ij}.

3. Nonbasic variable X_{ij}: These equal arc(i, j)'s lower bound L_{ij}. p:

依據上述準則一般而言, n node, n-1 node (arc)
如能形成(基本變數) spanning tree 即是 (connecting all nodes, 沒有 cycle. here 不需 form a path)





小型網路 以試誤法即可完成.

step 1. Determine a starting bfs. The n-1 basic variables will corresponding to a spanning tree. Indicate the nonbasic variables at their upper bound by dashed arcs.

step 2. Compute $y_1, y_2, \ldots y_n$ (simplex multipliers) by solving $y_1 = 0$ $y_i - y_j = c_{ij}$ for all basic variables

x_{ij}. For all nonbasic variables, determine the row 0 coefficient $\bar{C}_{ij}$ from $\bar{C}_{ij} = y_i - y_j - C_{ij}$. The current bfs is optimal $\bar{C}_{ij} \le 0$, for all $x_{ij} = L_{ij}$ and $\bar{C}_{ij} \ge 0$ for all $x_{ij} = U_{ij}$. If the bfs is not optimal, choose the nonbasic variable that most violates the optimality conditions as the entering basic variables.

基本变数

$y_1 = 0 \quad y_1 - y_3 = \overset{\text{cost}}{12} \quad y_2 - y_5 = 6 \quad y_3 - y_5 = 7 \quad y_4 - y_5 = 3$

$\Rightarrow y_1 = 0, \quad y_3 = -12, \quad y_2 = -13, \quad y_5 = -19, \quad y_4 = -16$

非基本变数

$\bar{C}_{12} = y_1 - y_2 - c_{12} = 0 - (-13) - 10 = 3$ (满足 optimality condition nonbasic variables at upper bound $x_{ij} =$ $x_{ij} =$)

$\bar{C}_{14} = y_1 - y_4 - c_{14} = 0 - (-16) - 6 = 10$

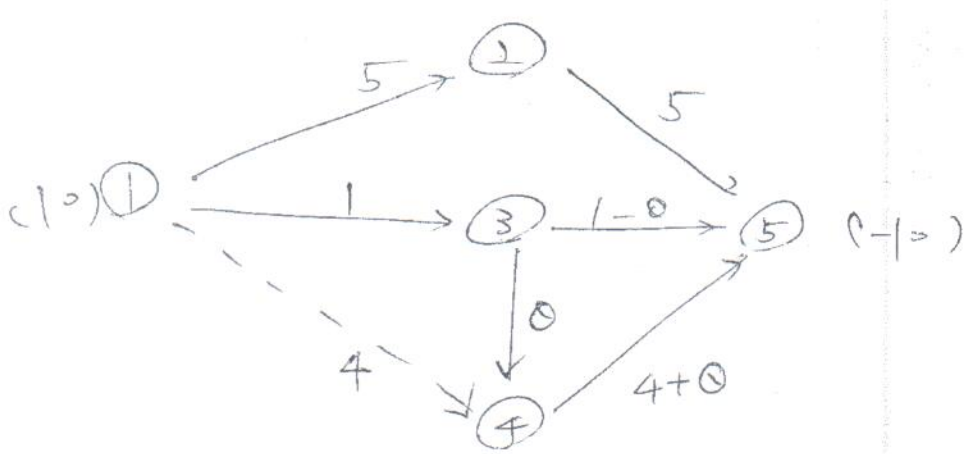
$\bar{C}_{32} = y_3 - y_2 - c_{32} = -12 - (-13) - 2 = -1$ (at lower bound $x_{ij} = 0$)

$\bar{C}_{34} = y_3 - y_4 - c_{34} = -12 - (-16) - 3 = 1$ (Violate optimality condition for nonbasic variable at lower bound)

$\bar{C}_{ij} > 0$. $x_{ij} = L_{ij}$ fi

Since $\overline{c_{34}} = 1 > 0$. 增加一单位 x_{34} (at its lower bound, it's OKay to increase it) 会减少 Z 一单位。於是可以改善 Z 值 選 x_{34} 為進入基本变数.

Step 3. Identify the cycle (there will be exactly one!) created by adding the arc corresponding to the entering variable to the current spanning tree of the current bfs. Use conservation of flow to determine the new values of the variables in the cycle. The variable that exits the basis will be the variable that first hits its upper bound or lower bound as the value of the entering.



3→4 最多增加 θ.

4→5 则变成 4+θ.

node 5 為維持流量均衡

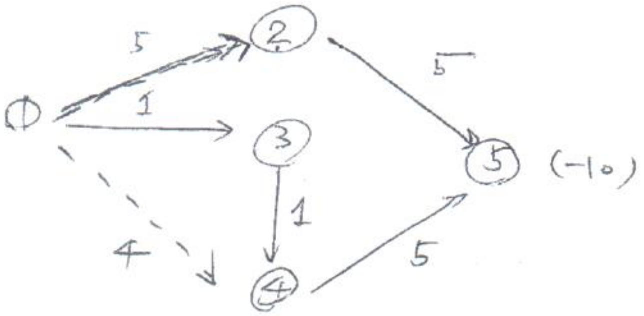
⇒ 3→5 為 1−θ.

$\theta \le 5 \quad U_{34} = 5.$

$4+\theta \le 6 \quad \theta \le 2.$

$1-\theta \ge 0.$

$\therefore \theta = 1$.



Step 4. Find the new bfs by changing the flows of the arcs in the cycle found in Step 3. Now go to Step 2.

$y_1 = 0 \quad y_1 - y_3 = 12 \quad y_3 - y_4 = 3 \quad y_2 - y_5 = 6 \quad y_4 - y_5 = 3$

$\Rightarrow y_1 = 0 \quad y_2 = -12, \quad y_3 = -12, \quad y_4 = -15, \quad y_5 = -18$

non basic variables $= U_{ij}$

$\bar{C}_{12} = y_1 - y_2 - C_{12} = 0 - (-12) - 10 = 2$ $\quad \bar{C}_{ij} \geq 0$

$\bar{C}_{14} = y_1 - y_4 - C_{14} = 0 - (-15) - 6 = 9$ $\quad \bar{C}_{ij} \geq 0$ satisfy optimality condition at upper bound

$\bar{C}_{32} = y_3 - y_2 - C_{32} = -12 - (-12) - 2 = -2$ $\quad \bar{C}_{ij} < 0$

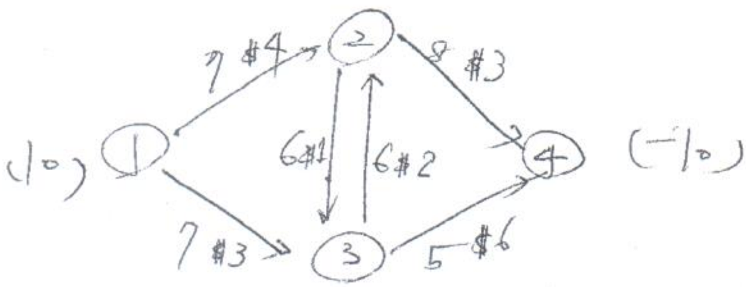
$\bar{C}_{35} = y_3 - y_5 - C_{35} = -12 - (-18) - 7 = -1$ $\quad \bar{C}_{ij} \leq 0$ at it low bound.

$x_{12} = 5, \quad x_{14} = 4$ (Upper bound variables)

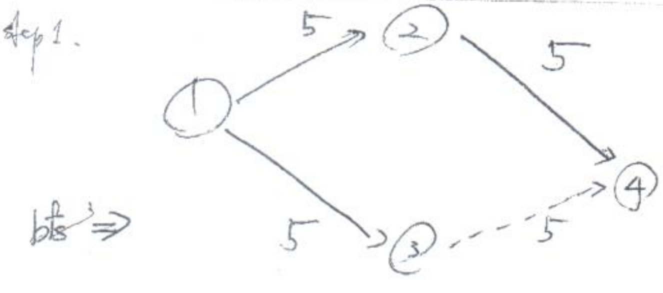
Lower bound Variable $x_{32} = x_{35} = 0$.

BASIC variables

$x_{13} = 1, x_{34} = 1, \quad x_{25} = 5, x_{45} = 5. \quad \underline{Z = 10}$ p5 e.g. Use the network simplex to solve MCNFP.



step 1.



$y_1 = 0$

$y_1 - y_2 = 4$

$y_1 - y_3 = 3$

$y_2 - y_4 = 3$

step 2.

$\Rightarrow \quad y_1 = 0 \quad y_2 = -4 \quad y_3 = -3 \quad y_4 = -7.$

$\bar{C}_{34} = -3-(-7)-6 = -2$

$\bar{C}_{23} = -4-(-3)-1 = -2$

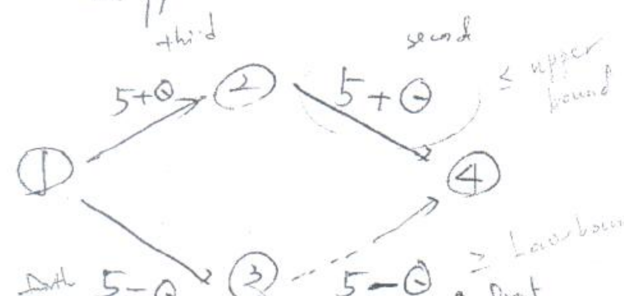
$\bar{C}_{32} = -3-(-4)-2 = -1$

$C_{ij} < 0.$ $\quad \underline{x_{ij} = L_{ij}}$ optimal $\quad \underline{x_{ij} = 5}$ 违反

$\Leftarrow$ $\boxed{\begin{array}{l} x_{ij} = 0 \\ x_{ij} = 0 \end{array} \text{满足最佳條件}}$

<u>x_{34} as entering</u>

once是 upper bound.



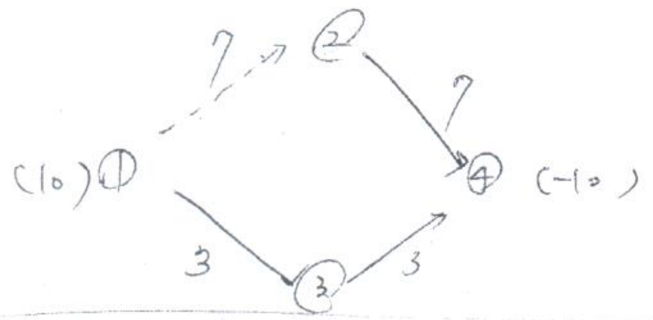
$1 \to 2 \quad 5+\theta \le 7$

$2 \to 4 \quad 5+\theta \le 8$

$1 \to 3 \quad 5-\theta > 0 \Rightarrow \theta < 5.$

$3 \to 4 \quad 5-\theta > 0$

$\longrightarrow \quad \theta = 2$



$y_1 = 0 \quad y_1 - y_3 = 3 \quad y_3 - y_4 = 6 \quad y_2 - y_4 = 3$

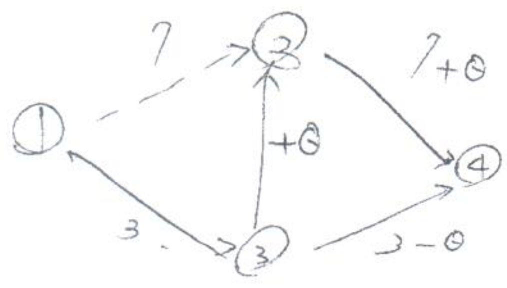
$\Rightarrow y_1 = 0 \quad y_2 = -6, \quad y_3 = -3, \quad y_4 = -9.$

$\bar{C}_{12} = 0 - (-6) - 4 = 2$ $\quad C_{ij} > 0 \; x_{ij} = U_{ij}$ optimal

$\bar{C}_{23} = -6 - (-3) - 1 = -4$ $\quad C_{ij} < 0, \; x_{ij} = L_{ij} = 0$ optimal

$\bar{C}_{32} = -3 - (-6) - 2 = 1$ $\quad C_{ij} > 0 \; \underline{\underline{x_{ij} = U_{ij}}}$ optimal

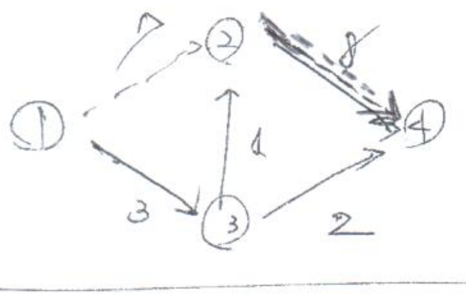
$x_{ij} = 0.$ [illegible]



$\underline{2 \to 4} \quad 7 + \theta \le 8$

$\underline{\underline{3 \to 4}} \quad 3 - \theta \ge 0$

$3 \to 2 \quad \theta \le 6 \Rightarrow \underline{\theta = 1}$



$y_1 = 0 \quad y_1 - y_3 = 3 \quad y_3 - y_2 = 2 \quad y_3 - y_4 = 6$

$\Rightarrow y_1 = 0 \quad y_2 = -5 \quad y_3 = -3. \quad y_4 = -9$

$\overline{C_{23}} = -5-(-3)-1 = -3 \quad < 0. \quad X_{ij} = L_{ij} = 0$

$\overline{C_{12}} = 0-(-5)-4 = 1$ (satisfy optimality condition)

$\overline{C_{24}} = -5-(-9)-3 = 1 \quad > 0. \quad X_{ij} = U_{ij}$

Basic Variable $x_{13} = 3$, $x_{32} = 1$, $x_{34} = 2$

Nonbasic variables at upper bound $x_{12} = 7$. $x_{24} = 8$

" lower bound $x_{23} = 0$.

$Z = 7(4) + 3(3) + 1(2) + 8(3) + 2(6) = 75$