Ch 7 隨机乱数產生

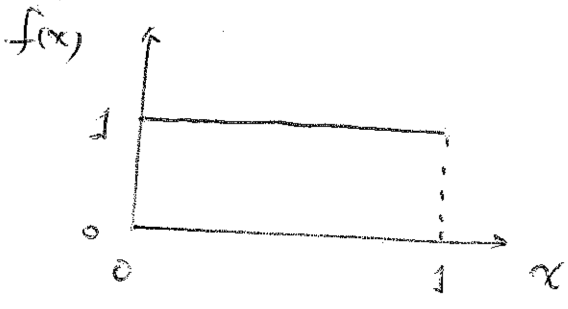
隨机乱数是所有離散系統的基本成份,大部分的電腦程式語言有副程式可產生隨机乱数。模擬語言亦有類似功能,用以產生隨机乱数可用來代表事件的時间与隨机乱数。本章介紹隨机乱数的產生与測試其隨机性,第八章介紹隨机乱数如何被用來產生代表許多分配之隨机變数。

7.1 隨机乱数之特性

一序列的隨机乱数 $R_1, R_2, \ldots$ 一定有兩項統計特性,一致性与独立。每一隨机乱数 R_i 是一從 $(0,1)$ 间連續性一致分配所抽取的独立樣本,也就是 p.d.f 為

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

其 p.d.f 如下圖。



期望值 $E(R)=\int_0^1 f(x)\,dx=\int_0^1 x\,dx=\frac{x^2}{2}\Big|_0^1=\frac{1}{2}$

變異數 $\int_0^1 x^2 dx-[E(R)]^2=\frac{x^3}{3}\Big|_0^1-\left(\frac{1}{2}\right)^2=\frac{1}{12}$

一些一致性與独立性質如下：

1. 如果(0,1)分成n組，在每組的期望觀察次數為 $\frac{N}{n}$，其中N為所有觀察數目。
2. 在一区間內觀察到某一數值之机率與前一抽取之數值独立。

7.2 Generation of Pseudo-Random Numbers

本節標題"Pseudo"代表假的，冒充的。Pseudo是用以代表隨机乱數的產生是使用一種方法，而這些隨机乱數不是真正的隨机，因為在一定長度之後，便會重覆。產生隨机乱數的目的儘量在(0,1)之間產生數字，具有一致性與独立性。

在產生pseudo-random numbers時，一些問題會產生，舉例如下：

1. 產生的數字不是一致分配
2. 產生的數字是離散的而不是連續的
3. 產生數字的平均數偏高或偏低
4. 產生數字的變異數偏高或偏低
5. 產生的數字不是獨立的，例如自我相關，
   數字持續上升或下降，許多數字低於平均數接著許多數字高於平均數。

在選擇產生隨機亂數的副程式時，應注意下列考量：

1. 速度應快。 2. 可攜性，便於不同電腦執行
3. 有足夠長的循環 4. 可以複製 5. 符合一致性與獨立性

7.3 Techniques for Generating Random Numbers

7.3.1 Linear Congruential Method 線性一致方法，最為廣泛使用的方法，

$$X_{i+1} = (aX_i + c) \bmod m \quad i=0,1,2,\ldots \quad (7.1)$$

餘數

$c \neq 0$ 稱為混合一致方法 (Mixed congruential method)

$c=0$ 稱為 乘法一致方法 (Multiplicative congruential method

選擇 a、c、m 和 X_0 將會影響循環的長度。

e.g. 7.1 令 $X_0=27$ $a=17$ $c=43$ $m=100$

會要創造的變數在 0 與 (100−1) 之間

隨机亂數在 (0,1) 之間可藉由

$$R_i = \frac{X_i}{m} \quad i=1,2,\ldots \qquad (7.2)$$

$$\begin{aligned}
X_0 &= 27 \\
X_1 &= (17 \cdot 27 + 43) \bmod 100 = 502 \bmod 100 = 2 \\
R_1 &= \frac{2}{100} = 0.02 \\
X_2 &= (17 \cdot 2 + 43) \bmod 100 = 77 \bmod 100 = 77 \\
R_2 &= \frac{77}{100} = 0.77 \\
*[3pt] X_3 &= (17 \cdot 77 + 43) \bmod 100 = 1352 \bmod 100 = 52 \\
R_3 &= \frac{52}{100} = 0.52 \\
&\vdots
\end{aligned}$$

$m=2^{31}-1$ 與 2^{48} 是模擬常用的數字。

Maximum density: the values assumed by R_i, leave no gaps on [0,1]

Maximum period: 可藉由適當選擇 a、c、m 和 X。

- m 是 2 的乘冪、例如 $m=2^b$, $c\neq0$

最長可能期間 $P=m=2^b$，可經由 c 是相對於 m 之質數，且 $a=1+4k$，k 是整數。

- m 是 2 的乘冪，例如 $m=2^b$，且 $c=0$。
  最長可能期間 $P=\frac{m}{4}=2^{b-2}$，可經由 X_0 是奇數 $a=3+8K$ 或 $a=5+8K$ 達成
- m 是質數和 $c=0$，最長可能期間 $P=m-1$，
  可經由 a^K-1 可被 m 除，$K=m-1$。

example 7.2

使用 multiplicative congruential method 找出亂數期間

$a=13$ $m=2^6=64$ $X_0=1, 2, 3,$ 和 4. 其解答在

Table 7.1

Table 7.1. Period Determination Using Various Seeds

i	X_i	X_i	X_i	X_i
0	1	2	3	4
1	13	26	39	52
2	41	18	59	36
3	21	42	63	20
4	17	34	51	4
5	29	58	23	
6	57	50	43	
7	37	10	47	
8	33	2	35	
9	45		7	
10	9		27	
11	53		31	
12	49		19	
13	61		55	
14	25		11	
15	5		15	
16	1		3	

由解答可看出 $X_0=1$, $X_0=3$ 時期間最長為 16。

由 $P=\frac{m}{4}=2^{b-2}=\frac{64}{4}=16$

可知最長期間 16。

$a=3+8K$ 或 $5+8K$

今 $a=13$ $\therefore K=1$。

這例子只是示範，週期時間太短無法應用

詳細選擇 a、c、m 和 X_0 非常重要。

EXAMPLE 7.3

Let $m = 10^2 = 100$, $a = 19$, $c = 0$, and $X_0 = 63$, and generate a sequence of random integers using Equation (7.1).

$$
\begin{aligned}
X_0 &= 63 \\
X_1 &= (19)(63) \bmod 100 &&= 1\underline{197} \bmod 100 = 97 \\
X_2 &= (19)(97) \bmod 100 &&= 18\underline{43} \bmod 100 = 43 \\
X_3 &= (19)(43) \bmod 100 &&= 8\underline{17} \bmod 100 = 17 \\
&\vdots
\end{aligned}
$$

10的乘幂保留百位以下

When m is a power of 10, say $m = 10^b$, the modulo operation is accomplished by saving the b rightmost (decimal) digits. By analogy, the modulo operation is most efficient for binary computers when $m = 2^b$ for some $b > 0$. 數字

✓example 7.4 介绍一廣泛被測試過的方法

$a = 7^5 = 16807$ $m = 2^{31} - 1 = 2147483647$ (質數)

$C = 0$ $X_0 = 123{,}457$

numbers generated are as follows:

$$
\begin{aligned}
X_1 &= 7^5(123{,}457) \bmod (2^{31} - 1) = 2{,}074{,}941{,}799 \bmod (2^{31} - 1) \\
X_1 &= 2{,}074{,}941{,}799 \\
R_1 &= \frac{X_1}{2^{31}} = 0.9662 \\
X_2 &= 7^5(2{,}074{,}941{,}799) \bmod (2^{31} - 1) = 559{,}872{,}160 \\
R_2 &= \frac{X_2}{2^{31}} = 0.2607 \\
X_3 &= 7^5(559{,}872{,}160) \bmod (2^{31} - 1) = 1{,}645{,}535{,}613 \\
R_3 &= \frac{X_3}{2^{31}} = 0.7662 \\
&\vdots
\end{aligned}
$$

7.3.2 Combined Linear Congruential Generators

一有用的方法是合併兩个或兩个以上 multiplicative congruential generators

$$X_i = \left(\sum_{j=1}^{k} (-1)^{j-1} X_{i,j} \right) \bmod m_1 - 1$$

$$R_i = \begin{cases} \dfrac{X_i}{m_1} & X_i > 0 \\ \dfrac{m_1 - 1}{m_1}, & X_i = 0 \end{cases}$$

$(-1)^{j-1}$ 代表 "+" "−"

最大可能期間

$$P = \frac{(m_1-1)(m_2-1)\cdots(m_k-1)}{2^{k-1}}$$

EXAMPLE 7.5

For 32-bit computers, L'Ecuyer [1988] suggests combining $k = 2$ generators with $m_1 = 2147483563$, $a_1 = 40014$, $m_2 = 2147483399$, and $a_2 = 40692$. This leads to the following algorithm:

1. Select seed $X_{1,0}$ in the range $[1, 2147483562]$ for the first generator, and seed $X_{2,0}$ in the range $[1, 2147483398]$.
   Set $j = 0$.
2. Evaluate each individual generator.
$$X_{1,j+1} = 40014 X_{1,j} \bmod 2147483563$$
$$X_{2,j+1} = 40692 X_{2,j} \bmod 2147483399$$
3. Set
$$X_{j+1} = (X_{1,j+1} - X_{2,j+1}) \bmod 2147483562$$
4. Return
$$R_{j+1} = \begin{cases} \dfrac{X_{j+1}}{2147483563}, & X_{j+1} > 0 \\[2ex] \dfrac{2147483562}{2147483563}, & X_{j+1} = 0 \end{cases}$$
5. Set $j = j+1$ and go to step 2.

L'Ecuyer 的期間有 $\dfrac{(m_1-1)(m_2-1)}{2} \approx 2 \times 10^{18}$

7.4 Tests for Random Numbers

7.1 討論的隨机亂數特性 一致性与独立性 為確定這兩項特性可以滿足，一些檢定可供使用且与軟体結合。Frequency test 是檢定一致性，Runs test, Autocorrelation test, Gap test, Poker test 是檢定独立性。

1. Frequency test — 使用 Kolmogorov-Smirnov 或 χ^2 test 檢定資料一致性

2. Runs test — 連串檢定藉著比較資料高於或低於平均值，統計量與 χ^2 比較。

3. Autocorrelation test — 檢定數目間的相關性。

4. Gap test — 計算在一特定數字在不同運算間差出次數，並使用 K-S 檢定比較期望的缺口。

5. Poker test — 將數目分群如手中的 Poker，並使用 χ^2 test 比較其期望值。

7.4.1 Frequency Tests

1. The Kolmogorov-Smirnov test：

比較一致分配的 c.d.f $F(x)$ 與樣本的實証 c.d.f，$S_N(x)$。其中 N 代表樣本數目。

由定義

$$F(x) = x, \quad 0 \le x \le 1$$

如果隨机樣本由隨机亂數產生器產生

$R_1, R_2, \ldots, R_N$ 則實証 c.d.f.

$$S_N(x) = \frac{\text{number of } R_1, R_2, \ldots, R_N \text{ which are} \le x}{N}$$

當 N 変大, $S_N(x)$ 應該是对 $F(x)$ 好的近似.

K-S 檢定是基於

$$D = \max \left| F(x) - S_N(x) \right|$$

樣本分配 of D 統計量, 可由表 A.8 查出

其測試步驟如下

Step 1. Rank the data from smallest to largest. Let $R_{(i)}$ denote the ith smallest observation, so that

$$R_{(1)} \le R_{(2)} \le \cdots \le R_{(N)}$$

Step 2. Compute

$$D^+ = \max_{1 \le i \le N} \left\{ \frac{i}{N} - R_{(i)} \right\}$$

$$D^- = \max_{1 \le i \le N} \left\{ R_{(i)} - \frac{i-1}{N} \right\}$$

Step 3. Compute $D = \max(D^+, D^-)$.

Step 4. Determine the critical value, D_α, from Table A.8 for the specified significance level α and the given sample size N.

Step 5. If the sample statistic D is greater than the critical value, D_α, the null hypothesis that the data are a sample from a uniform distribution is rejected. If $D \le D_\alpha$, conclude that no difference has been detected between the true distribution of $\{R_1, R_2, \ldots, R_N\}$ and the uniform distribution.

example 7.6 假設 5 个數字 0.44, 0.81, 0.14, 0.05, 0.93 被產生, 欲使用 K-S 檢定在 $\alpha = .05$ 情形

下檢定資料的一致性

Table 7.2 顯示 K-S 檢定

$R_{(i)}$	0.05	0.14	0.44	0.81	0.93
i/N	0.2	0.4	0.6	0.8	1
$i/N - R_{(i)}$	0.15	0.26	0.16	—	0.07
$R_{(i)} - \frac{(i-1)}{N}$	0.05	—	0.04	0.21	0.13

首先將 $R_{(i)}$ 由小排到大(第一列)

再來計算 D^+ 與 D^- 亦即第三列與第四列

$$D^+ = \max_{1 \le i \le N} \left\{ \frac{i}{N} - R_{(i)} \right\} = 0.26$$

$$D^- = \max_{1 \le i \le N} \left\{ R_{(i)} - \frac{i-1}{N} \right\} = 0.21$$

step3: $D = \max(D^+, D^-) = 0.26$

step4: 查表 $\alpha = .05$ $N = 5$ $D_\alpha = 0.565$

step5: if $D \le D_\alpha$ conclude that no difference has been detected between the true distribution of $\{R_1, R_2, \ldots, R_N\}$ and uniform distribution.

既然 $0.26 < D_\alpha = 0.565$，接受一致性檢定

2. χ^2 test:

$$\chi_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

O_i：第 i 組觀察次數

E_i：第 i 組期望觀察次數　$E_i = \frac{N}{n}$

N 是全部觀察次數，n：是分組組數

χ_0^2 大致服從 χ^2 分配自由度 $n-1$

Example 7.7

0.34	0.90	0.25	0.89	0.87	0.44	0.12	0.21	0.46	0.67
0.83	0.76	0.79	0.64	0.70	0.81	0.94	0.74	0.22	0.74
0.96	0.99	0.77	0.67	0.56	0.41	0.52	0.73	0.99	0.02
0.47	0.30	0.17	0.82	0.56	0.05	0.45	0.31	0.78	0.05
0.79	0.71	0.23	0.19	0.82	0.93	0.65	0.37	0.39	0.42
0.99	0.17	0.99	0.46	0.05	0.66	0.10	0.42	0.18	0.49
0.37	0.51	0.54	0.01	0.81	0.28	0.69	0.34	0.75	0.49
0.72	0.43	0.56	0.97	0.30	0.94	0.96	0.58	0.73	0.05
0.06	0.39	0.84	0.24	0.40	0.64	0.40	0.19	0.79	0.62
0.18	0.26	0.97	0.88	0.64	0.47	0.60	0.11	0.29	0.78

χ^2 在樣本大時較有效，例如 $N > 50$.

Table 7.3. Computations for Chi-Square Test

分組組界	*Interval*	O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
0.01 - 0.1	1	8	10	−2	4	0.4
0.11 - 0.2	2	8	10	−2	4	0.4
0.21 - 0.3	3	10	10	0	0	0.0
0.31 - 0.4	4	9	10	−1	1	0.1
0.41 - 0.5	5	12	10	2	4	0.4
0.51 - 0.6	6	8	10	−2	4	0.4
0.61 - 0.7	7	10	10	0	0	0.0
0.71 - 0.8	8	14	10	4	16	1.6
0.81 - 0.9	9	10	10	0	0	0.0
0.91 - 1.00	10	11	10	1	1	0.1
		100	100	0		3.4

$k - s - 1 = \underset{\text{分組數}}{10} - \underset{\text{參數個數}}{1} - 1$

$\chi_0^2 = 3.4$　　$\chi^2_{.05,8} = 15.5$　　查表 A.6

既然 $\chi_0^2 < \chi^2_{.05,8}$ 接受 H_0 (

7.4.2 Run Tests 連串檢定

前面介紹的 K-S 檢定與 χ^2 檢定是用以測試資料的一致性。連串檢定是用以檢定資料的隨機性(獨立性)。

連串被定義成一連串事件 相似 ，以擲銅板為例

H T T H H T T T H T

Head：為一相同事件　Tail：尾：為一相同事件

∴ 在上述結果有 6 個連串。

1. Runs up and runs down 上下連串法

0.08　0.18　0.23　0.36　0.42　0.55　0.63　0.72　0.89　0.91

— 基準　+　+　+　+　+　+　+　+　+

0.08　0.93　0.15　0.96　0.26　0.84　0.28　0.79　0.36　0.57

— 基準　+　−　+　−　+　−　+　−　+

上述例子是太少或太多連串。

後面一組是up down

0.08　0.68　0.89　0.84　0.74　0.91　0.55　0.71　0.36

常用 ✓ 2種方式　+　+　−　−　+　−　+　−

— 基準　+　+　−　−　+　−　+　−

$$\mu_a = \frac{2N-1}{3} \qquad \sigma_a^2 = \frac{16N-29}{90}$$

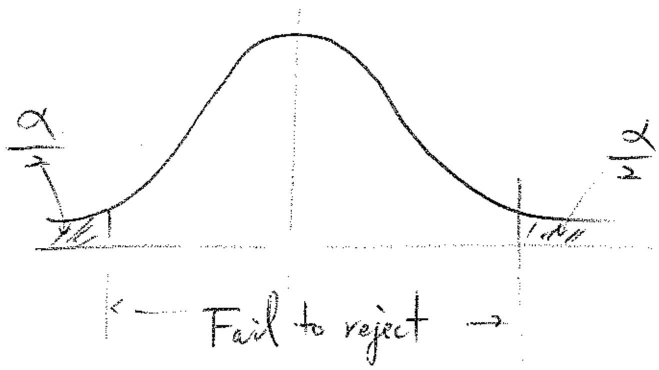
μ_a: 連串數目期望值

σ_a^2: 連串數目變異數

當 $N > 20$, 連串數目 $\sim N(\mu_a, \sigma_a^2)$ 約成常態分配

$$Z_0 = \frac{a - \mu_a}{\sigma_a}$$

$$Z_0 = \frac{a - \frac{2N-1}{3}}{\sqrt{\frac{16N-29}{90}}}$$



$Z_0 \sim N(0,1)$ 當 $-Z_{\frac{\alpha}{2}} \le Z_0 \le Z_{\frac{\alpha}{2}}$ Fail to reject H_0

EXAMPLE 7.8

Based on runs up and runs down, determine whether the following sequence of 40 numbers is such that the hypothesis of independence can be rejected where $\alpha = 0.05$.

0.41	0.68	0.89	0.94	0.74	0.91	0.55	0.62	0.36	0.27
0.19	0.72	0.75	0.08	0.54	0.02	0.01	0.36	0.16	0.28
0.18	0.01	0.95	0.69	0.18	0.47	0.23	0.32	0.82	0.53
0.31	0.42	0.73	0.04	0.83	0.45	0.13	0.57	0.63	0.29

The sequence of runs up and down is as follows:

基準

+ + + − + − + − − − + + − + − − + − +
− − + − − + − + + − − + + − + − − + +

There are 26 runs in this sequence. With $N = 40$ and $a = 26$, Equations (7.4) and (7.5) yield

在上述序列中有26個連串, $N=40$, $a=26$

$$\mu_a = \frac{2(40)-1}{3} = 26.33$$

$$\sigma_a^2 = \frac{16 \cdot 40 - 29}{90} = 6.79$$

$$Z_0 = \frac{26-26.33}{\sqrt{6.79}} = -0.13$$

$Z_{.025} = 1.96$ $\quad$ $Z_0 = -0.13 < Z_{.025} = 1.96$

不可拒絕 H_0

2. 平均數法（亦有中位數法與此法類似）

0.40	0.84	0.75	0.18	0.13	0.92	0.57	0.77	0.30	0.71
0.42	0.05	0.78	0.74	0.68	0.03	0.18	0.51	0.10	0.37

mean = 0.5

The pluses and minuses are as follows:

− + + − − + + + − + − − + + + − − + − −

與平均數比較，低於平均數給"−"；高於平均數給"+"，在資料中有11個連串，其中5個高於平均數，n_1，低於平均數有6個，n_2。

$$\mu_b = \frac{2n_1n_2}{N} + \frac{1}{2} \qquad \sigma_b^2 = \frac{2n_1n_2(2n_1n_2-N)}{N^2(N-1)}$$

$$Z_0 = \frac{b - \left(\frac{2n_1n_2}{N}\right) - \frac{1}{2}}{\sqrt{\frac{2n_1n_2(2n_1n_2-N)}{N^2(N-1)}}}$$

在本例中 $\mu_b = \frac{2 \cdot 5 \cdot 6}{20} + \frac{1}{2} = 3.5$

$$\sigma_b^2 = \frac{2 \cdot 5 \cdot 6 \, (2 \cdot 5 \cdot 6 - 20)}{(20)^2 (20-1)} = \frac{2400}{7600} = 0.3157$$

$\sigma_b = 0.5619$

$$Z_0 = \frac{11 - 3.5}{0.5619} = \frac{7.5}{0.5619} = 13.34$$

$$-1.96 = -Z_{.025} \le Z_0 \le Z_{.025} = 1.96$$

既然 $Z_0 = 13.34 > Z_{.025}$.

拒绝 H_0

EXAMPLE 7.9

Determine whether there is an excessive number of runs above or below the mean for the sequence of numbers given in Example 7.8. The assignment of +'s and −'s results in the following:

− + + + + + + + − − − + + − + − − − − −
− − + + − − − − + + − − + − + − − + + −

The values of n_1, n_2, and b are as follows:

$$n_1 = 18$$
$$n_2 = 22$$
$$N = n_1 + n_2 = 40$$
$$b = 17$$

Equations (7.6) and (7.7) are used to determine μ_b and σ_b^2 as follows:

$$\mu_b = \frac{2(18)(22)}{40} + \frac{1}{2} = 20.3$$

and

$$\sigma_b^2 = \frac{2(18)(22)[(2)(18)(22) - 40]}{(40)^2(40-1)} = 9.54$$

Since n_2 is greater than 20, the normal approximation is acceptable, resulting in a Z_0 value of

$$Z_0 = \frac{17 - 20.3}{\sqrt{9.54}} = -1.07$$

Since $z_{0.025} = 1.96$, the hypothesis of independence cannot be rejected on the basis of this test. ◀

另一項關心的是連串的長度(資料的個數)

$\overset{-}{0.16}$ $\overset{-}{0.27}$ $\overset{+}{0.58}$ $\overset{+}{0.63}$ $\overset{-}{0.45}$ $\overset{-}{0.21}$ $\overset{+}{0.72}$ $\overset{+}{0.87}$ $\overset{-}{0.27}$ $\overset{-}{0.15}$ $\overset{+}{0.92}$ $\overset{+}{0.85}$.

如果資料如上述,使用上述介紹的方法檢定其獨立性,資料會接受 H_0。但是每一連串的長度均為 2,並不是期望的結果。

令 Y_i 為連串長度(資料個數)i 的數目

(上下法)

$$E(Y_i) = \frac{2}{(i+3)!}\left[N(i^2+3i+1)-(i^3+3i^2-i-4)\right],\ i \le N-2$$

$$E(Y_i) = \frac{2}{N!},\quad i = N-1$$

其中 N 為觀察值數目

$E(Y_i)$ 可由下列近似(平均數法)

$$E(Y_i) = \frac{N\,W_i}{E(I)},\ N > 20$$

$$W_i = \left(\frac{n_1}{N}\right)^i\left(\frac{n_2}{N}\right) + \left(\frac{n_1}{N}\right)\left(\frac{n_2}{N}\right)^i,\ N > 20$$

W_i:連串有長度 i 的近似機率

$$E(I) = \frac{n_1}{n_2} + \frac{n_2}{n_1},\ N > 20$$

$E(I)$:連串之期望長度

$$\chi_0^2 = \sum_{i=1}^{L} \frac{[O_i - E(Y_i)]^2}{E(Y_i)}$$

$L = N-1$ up and down

$L = N$ mean

example 7.10

Given the following sequence of numbers, can the hypothesis that the numbers are independent be rejected on the basis of the length of runs up and down at $\alpha = 0.05$?

up and down

0.30	0.48	0.36	0.01	0.54	0.34	0.96	0.06	0.61	0.85
0.48	0.86	0.14	0.86	0.89	0.37	0.49	0.60	0.04	0.83
0.42	0.83	0.37	0.21	0.90	0.89	0.91	0.79	0.57	0.99
0.95	0.27	0.41	0.81	0.96	0.31	0.09	0.06	0.23	0.77
0.73	0.47	0.13	0.55	0.11	0.75	0.36	0.25	0.23	0.72
0.60	0.84	0.70	0.30	0.26	0.38	0.05	0.19	0.73	0.44

For this sequence the +'s and −'s are as follows:

+ − − + − + − + + − + − + + − + + − +
− + − − + − + − − + − − + + + − − − + +
− − − + − + − − − + − + − − − + − + + −

The length of runs in the sequence is as follows:

1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 2, 1, 1, 1, 1, 2, 1, 1,
1, 2, 1, 2, 3, 3, 2, 3, 1, 1, 1, 3, 1, 1, 1, 3, 1, 1, 2, 1

The number of observed runs of each length is as follows:

Run Length, i	1	2	3
Observed Runs, O_i	26	9	5

The expected numbers of runs of lengths one, two, and three are computed from Equation (7.8) as

$$E(Y_1) = \frac{2}{4!}[60(1 + 3 + 1) - (1 + 3 - 1 - 4)] = 25.08$$

$$E(Y_2) = \frac{2}{5!}[60(4 + 6 + 1) - (8 + 12 - 2 - 4)] = 10.77$$

$$E(Y_3) = \frac{2}{6!}[60(9 + 9 + 1) - (27 + 27 - 3 - 4)] = 3.04$$

The mean total number of runs (up and down) is given by Equation (7.4) as

$$\mu_a = \frac{2(60) - 1}{3} = 39.67$$

Thus far, the $E(Y_i)$ for $i = 1, 2$, and 3 total 38.89. The expected number of runs of length 4 or more is the difference $\mu_a - \sum_{i=1}^{3} E(Y_i)$, or 0.78.

As observed by Hines and Montgomery [1990], there is no general agreement regarding the minimum value of expected frequencies in applying the chi-square test. Values of 3, 4, and 5 are widely used, and a minimum of 5 was suggested earlier in this chapter. Should an expected frequency be too small,

Table 7.4 Length of Runs Up and Down: χ^2 Test

Run Length, i	Observed Number of Runs, O_i		Expected Number of Runs, $E(Y_i)$		$\frac{[O_i - E(Y_i)]^2}{E(Y_i)}$
1	26		25.08		0.03
2	9	14	10.77	14.59	0.02
≥ 3	5		3.82		
	40		39.67		0.05

it can be combined with the expected frequency in an adjacent class interval. The corresponding observed frequencies would then be combined also, and L would be reduced by one. With the foregoing calculations and procedures in mind, we construct Table 7.4. The critical value $\chi^2_{0.05,1}$ is 3.84. (The degrees of freedom equals the number of class intervals minus one.) Since $\chi_0^2 = 0.05$ is less than the critical value, the hypothesis of independence cannot be rejected on the basis of this test. ◀

EXAMPLE 7.11

Given the same sequence of numbers in Example 7.10, can the hypothesis that the numbers are independent be rejected on the basis of the length of runs above and below the mean at $\alpha = 0.05$? For this sequence, the +'s and −'s are as follows:

$$
\begin{array}{cccccccccccccccccccc}
- & - & - & - & + & - & + & - & + & + & - & + & - & + & + & - & - & + & - & + \\
- & + & - & - & + & + & + & + & + & + & + & - & - & + & + & - & - & - & - & + \\
- & - & - & + & - & + & - & - & - & + & + & + & + & - & - & - & - & - & + & -
\end{array}
$$

The number of runs of each length is as follows:

Run Length, i	1	2	3	≥ 4
Observed Runs, O_i	17	9	1	5

There are 28 values above the mean ($n_1 = 28$) and 32 values below the mean ($n_2 = 32$). The probabilities of runs of various lengths, w_i, are determined from Equation (7.11) as

$$\omega_i = \left(\frac{n_1}{N}\right)^i\left(\frac{n_2}{N}\right) + \left(\frac{n_1}{N}\right)\left(\frac{n_2}{N}\right)^i = \left(\frac{28}{60}\right)^1 \frac{32}{60} + \frac{28}{60}\left(\frac{32}{60}\right)^1 = 0.498$$

[illegible] formula

$$w_2 = \left(\frac{28}{60}\right)^2 \frac{32}{60} + \frac{28}{60}\left(\frac{32}{60}\right)^2 = 0.249$$

$$w_3 = \left(\frac{28}{60}\right)^3 \frac{32}{60} + \frac{28}{60}\left(\frac{32}{60}\right)^3 = 0.125$$

$$\vdots$$

$$E(I) = \frac{28}{32} + \frac{32}{28} = 2.02$$

Now, Equation (7.10) can be used to determine the expected numbers of runs of various lengths as

$$E(Y_i) = \frac{N w_i}{E(I)}$$

$$E(Y_1) = \frac{60(0.498)}{2.02} = 14.79$$

$$E(Y_2) = \frac{60(0.249)}{2.02} = 7.40$$

$$E(Y_3) = \frac{60(0.125)}{2.02} = 3.71$$

The total number of runs expected is given by Equation (7.13) as $E(A) = 60/2.02 = 29.7$. This indicates that approximately 3.8 runs of length four or more can be expected. Proceeding by combining adjacent cells in which $E(Y_i) < 5$ produces Table 7.5.

Table 7.5. Length of Runs Above and Below the Mean: χ^2 Test

Run Length, i	*Observed Number of Runs, O_i*	*Expected Number of Runs, $E(Y_i)$*	$\frac{[O_i - E(Y_i)]^2}{E(Y_i)}$
1	17	14.79	0.33
2	9	7.40	0.35
3	1 } 6	3.71 } 7.51	} 0.30
≥ 4	5	3.80	
	32	29.70	0.98

The critical value $\chi^2_{0.05,2}$ is 5.99. (The degrees of freedom equals the number of class intervals minus one.) Since $\chi_0^2 = 0.98$ is less than the critical value, the hypothesis of independence cannot be rejected on the basis of this test. ◀

7.4.3 自我相關檢定

7.4.3 Tests for Autocorrelation

The tests for autocorrelation are concerned with the dependence between numbers in a sequence. As an example, consider the following sequence of numbers:

0.12	0.01	0.23	0.28	0.89	0.31	0.64	0.28	0.83	0.93
0.99	0.15	0.33	0.35	0.91	0.41	0.60	0.27	0.75	0.88
0.68	0.49	0.05	0.43	0.95	0.58	0.19	0.36	0.69	0.87

From a visual inspection, these numbers appear random, and they would probably pass all the tests presented to this point. However, an examination of the 5th, 10th, 15th (every five numbers beginning with the fifth), and so on, indicates a very large number in that position. Now, 30 numbers is a rather small

$H_0: \rho_{im} = 0$ 　　i: ith number

$H_1: \rho_{im} \neq 0$ 　　m: lag

$$Z_0 = \frac{\hat{\rho}_{im}}{\sigma_{\hat{\rho}_{im}}}$$

M: 是 $i+(M+1)m \le N$ 的最大整數

N: 觀察值數目

$$\hat{\rho}_{im} = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+km} R_{i+(k+1)m}\right] - 0.25$$

$$\sigma_{\hat{\rho}_{im}} = \frac{\sqrt{13M+7}}{12(M+1)}$$

若 $-Z_{\frac{\alpha}{2}} \le Z_0 \le Z_{\frac{\alpha}{2}}$, do not reject H_0

$\rho_{im} > 0$ 正自我相關

$\rho_{im} < 0$ 負自我相關

Example 7.12

檢定 3rd, 8th, 13th, 18th, 23th, 28th 這些數字是自我相關的. 使用 $\alpha = .05$

在本例中 $i=3$, $m=5$（每5个資料）

$$M = i+(M+1)m \le N = 3+(M+1)\cdot 5 \le 30$$

$$\therefore M=4$$

則 $$\hat{\rho}_{35} = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+km}\cdot R_{i+(k+1)m}\right] - 0.25$$

$$= \frac{1}{4+1}\left[(0.23)(0.28)+(0.28)(0.33)+(0.33)(0.27)+(0.27)(0.05)+(0.05)(0.36)\right] - 0.25$$

$$= -0.1945$$

$$\sigma_{\hat{\rho}_{35}} = \frac{\sqrt{13M+7}}{12(M+1)} = \frac{\sqrt{13\cdot 4+7}}{12(4+1)} = 0.128$$

$$Z_0 = \frac{-0.1945}{0.128} = -1.516$$

$$Z_{0.025} = 1.96$$

既然 $-Z_{0.025} \le Z_0 \le Z_{0.025}$

接受 H_0

7.4.4 Gap Test

The gap test is used to determine the significance of the interval between the recurrences of the same digit. A gap of length x occurs between the recurrences of some specified digit. The following example illustrates the length of gaps associated with the digit 3:

$$
\begin{array}{l}
4,\ 1,\ \underline{3},\ 5,\ 1,\ 7,\ 2,\ 8,\ 2,\ 0,\ 7,\ 9,\ 1,\ \underline{3},\ 5,\ 2,\ 7,\ 9,\ 4,\ 1,\ 6,\ \underline{3} \\
\underline{3},\ 9,\ 6,\ \underline{3},\ 4,\ 8,\ 2,\ \underline{3},\ 1,\ 9,\ 4,\ 4,\ 6,\ 8,\ 4,\ 1,\ \underline{3},\ 8,\ 9,\ 5,\ 5,\ 7 \\
\underline{3},\ 9,\ 5,\ 9,\ 8,\ 5,\ \underline{3},\ 2,\ 2,\ \underline{3},\ 7,\ 4,\ 7,\ 0,\ \underline{3},\ 6,\ \underline{3},\ 5,\ 9,\ 9,\ 5,\ 5 \\
5,\ 0,\ 4,\ 6,\ 8,\ 0,\ 4,\ 7,\ 0,\ \underline{3},\ \underline{3},\ 0,\ 9,\ 5,\ 7,\ 9,\ 5,\ 1,\ 6,\ 6,\ \underline{3},\ 8 \\
8,\ 8,\ 9,\ 2,\ 9,\ 1,\ 8,\ 5,\ 4,\ 4,\ 5,\ 0,\ 2,\ \underline{3},\ 9,\ 7,\ 1,\ 2,\ 0,\ \underline{3},\ 6,\ 3
\end{array}
$$

To facilitate the analysis, the digit 3 has been underlined. There are eighteen 3's in the list. Thus, only 17 gaps can occur. The first gap is of length 10, the second gap is of length 7, and so on. The frequency of the gaps is of interest. The probability of the first gap is determined as follows:

$$
\begin{aligned}
P(\text{gap of } 10) &= \overbrace{P(\text{no } 3)\cdots P(\text{no } 3)}^{10 \text{ of these terms}}\,P(3) \\
&= (0.9)^{10}(0.1)
\end{aligned}
$$

since the probability that any digit is not a 3 is 0.9, and the probability that any digit is a 3 is 0.1. In general,

$$P(t \text{ followed by exactly } x \text{ non-}t \text{ digits}) = (0.9)^x(0.1), \quad x = 0, 1, 2, \ldots$$

In the example above, only the digit 3 was examined. However, to fully analyze a set of numbers for independence using the gap test, every digit, 0, 1, 2, ..., 9, must be analyzed. The observed frequencies of the various gap sizes for all the digits are recorded and compared to the theoretical frequency using the Kolmogorov-Smirnov test for discretized data.

The theoretical frequency distribution for randomly ordered digits is given by

$$\frac{a(1-r^n)}{r}$$

$$P(\text{gap} \le x) = F(x) = 0.1\sum_{n=0}^{x}(0.9)^n = 1 - 0.9^{x+1} \qquad (7.14)$$

$P(\text{gap} \le 5) = 0.1(0.9)^0 + 0.1(0.9)^1 + 0.1(0.9)^2 + \cdots + (0.1)(0.9)^5 = 0.1(1 + 0.9 + 0.9^2 + \cdots + 0.9^5) = 0.1\dfrac{1-0.9^{5+1}}{1-0.9} = 1 - 0.9^{5+1}$

The procedure for the test follows the steps below. When applying the test to random numbers, class intervals such as [0, 0.1), [0.1, 0.2), ... play the role of random digits.

Step 1. Specify the cdf for the theoretical frequency distribution given by Equation (7.14) based on the selected class interval width.

Step 2. Arrange the observed sample of gaps in a cumulative distribution with these same classes.

Step 3. Find D, the maximum deviation between $F(x)$ and $S_N(x)$ as in Equation (7.3).

Step 4. Determine the critical value, D_α, from Table A.8 for the specified value of α and the sample size N.

Step 5. If the calculated value of D is greater than the tabulated value of D_α, the null hypothesis of independence is rejected.

It should be noted that using the Kolmogorov-Smirnov test when the underlying distribution is discrete results in a reduction in the Type I error, α, and an increase in the Type II error, β. The exact value of α can be found using the methodology described by Conover [1980].

EXAMPLE 7.13

Based on the frequency with which gaps occur, analyze the 110 digits above to test whether they are independent. Use $\alpha = 0.05$. The number of gaps is given by the number of data values minus the number of distinct digits, or $110 - 10 = 100$ in the example. The number of gaps associated with the various digits are as follows:

Digit	0	1	2	3	4	5	6	7	8	9
Number of Gaps	7	8	8	17	10	13	7	8	9	13

以 computer programs 輔助

The gap test is presented in Table 7.6. The critical value of D is given by

4. Table A.8 $\frac{1.36}{\sqrt{n}} = D_{0.05} = \frac{1.36}{\sqrt{100}} = 0.136$

Since $D = \max|F(x) - S_N(x)| = 0.0224$ is less than $D_{0.05}$, do not reject the hypothesis of independence on the basis of this test. ◀

5. Since $D \le D_\alpha$ do not reject H_0

Table 7.6. Gap-Test Example

Gap Length	*Frequency*	*Relative Frequency*	*Cumulative Relative Frequency* $S_N(x)$	(7.14) $F(x)$	$\lvert F(x) - S_N(x)\rvert$
0–3	35	0.35	0.35	$1-0.9^4 =$ 0.3439	0.0061
4–7	22	0.22	0.57	$1-0.9^8 =$ 0.5695	0.0005
8–11	17	0.17	0.74	0.7176	0.0224
12–15	9	0.09	0.83	0.8147	0.0153
16–19	5	0.05	0.88	0.8784	0.0016
20–23	6	0.06	0.94	0.9202	0.0198
24–27	3	0.03	0.97	0.9497	0.0223
28–31	0	0.0	0.97	0.9657	0.0043
32–35	0	0.0	0.97	0.9775	0.0075
36–39	2	0.02	0.99	0.9852	0.0043
40–43	0	0.0	0.99	0.9903	0.0003
44–47	1	0.01	1.00	0.9936	0.0064

借助 Computer

歸納

例如 1 的 GAP length 為 45 有一次

上表之 Frequency 係

加總 0、1、2、3、4、5、6、7、8、9

之所有 GAP length 整理而成

7.4.5 Poker Test

The poker test for independence is based on the frequency with which certain digits are repeated in a series of numbers. The following example shows an unusual amount of repetition:

$$0.255, \quad 0.577, \quad 0.331, \quad 0.414, \quad 0.828, \quad 0.909, \quad 0.303, \quad 0.001, \quad \ldots$$

In each case, a pair of like digits appears in the number that was generated. In three-digit numbers there are only three possibilities, as follows:

1. The individual numbers can all be different.
2. The individual numbers can all be the same.
3. There can be one pair of like digits.

The probability associated with each of these possibilities is given by the following:

$$\begin{aligned} P(\text{three different digits}) &= P(\text{second different from the first}) \\ &\quad \times P(\text{third different from the first and second}) \\ &= (0.9)(0.8) = 0.72 \\ P(\text{three like digits}) &= P(\text{second digit same as the first}) \\ &\quad \times P(\text{third digit same as the first}) \\ &= (0.1)(0.1) = 0.01 \\ P(\text{exactly one pair}) &= 1 - 0.72 - 0.01 = 0.27 \end{aligned}$$

0.1 0.9 0.8

$10 \times (\square\,\square\,\square)$

1 1 1

2 3 4 5 6 7 8 9 0

Alternatively, the last result can be obtained as follows:

$$P(\text{exactly one pair}) = \binom{3}{2}(0.1)(0.9) = 0.27$$

P(second digit same as the first) P(third digit different from the first)

○○△
△○○
○△○

The following example shows how the poker test (in conjunction with the chi-square test) is used to ascertain independence.

EXAMPLE 7.14

A sequence of 1000 three-digit numbers has been generated and an analysis indicates that 680 have three different digits, 289 contain exactly one pair of like digits, and 31 contain three like digits. Based on the poker test, are these numbers independent? Let $\alpha = 0.05$. The test is summarized in Table 7.7.

The appropriate degrees of freedom are one less than the number of class intervals. Since $47.65 > \chi^2_{0.05,2} = 5.99$, the independence of the numbers is rejected on the basis of this test. ◀

Table 7.7. Poker-Test Results

Combination, i	*Observed Frequency, O_i*	*Expected Frequency, E_i*	$\frac{(O_i - E_i)^2}{E_i}$
Three different digits	680	720	2.22
Three like digits	31	10	44.10
Exactly one pair	289	270	1.33
	1000	1000	47.65