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APPLICATIONS OF SIMULATION MODELS IN FINANCE AND INSURANCE

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ABSTRACT

We describe a number of applications of simulation methods to practical problems in finance and insurance. The first entails the simulation of a two-stage model of a property-casualty insurance operation. The second application simulates the operation of an insurance regime for home equity conversion mortgages (also known as reverse mortgages). The third is an application of simulation in the context of *Value at Risk*, a widely-used measure for assessing the performance of portfolios of assets and/or liabilities. We conclude with an application of simulation in the testing of the *efficient market hypothesis* of the U.S. stock market.

1 FREQUENCY-SEVERITY INSURANCE MODEL

1.1 Formulating the Problem

In property and casualty insurance, as well as in health insurance, the actuary is often asked to predict the amount of insured losses during the next period of observation, such as a calendar year. In doing so, the actuary frequently has the results observed for a number of prior periods. Then if S_i is a random variable representing the amount of aggregate claims during the ith policy year (or, equivalently, the ith period of observation), the problem may be considered to be the estimation of the quantity

$$\Pr[S_{m+1} \le s_{m+1} \mid S_1, S_2, \ldots, S_m]$$

where $s_{m+1} \ge 0$. This is the conditional probability of the incurred losses during period $m+1$, given the results of the first m periods. Such a probability distribution is usually called a *predictive distribution*.

One way of approaching this problem is to first determine the distribution of the frequency of loss (i.e., the number of insurance claims) and then to determine the severity or amount of each individual claim.

We assume that given a parameter θ, the random variables $S_1, S_2, \ldots, S_{m+1}$ are independent and identically distributed with conditional probability density function p. We use f to denote the density function of θ. Thus, using Bayes' Theorem, we can write the conditional density function of S_{m+1}, given $S_1, S_2, \ldots, S_m$ as

$$\frac{\int p(s_{m+1} \mid \theta) \cdot \prod_{i=1}^{m} p(s_i \mid \theta) \cdot f(\theta) d\theta}{\int \prod_{i=1}^{m} p(s_i \mid \theta) f(\theta) d\theta},$$

where

$$\frac{\prod_{i=1}^{m} p(s_i \mid \theta) \cdot f(\theta)}{\int \prod_{i=1}^{m} p(s_i \mid \theta) \cdot f(\theta)}$$

is the posterior density function of θ, given $S_1 = s_1, S_2 = s_2, \ldots, S_m = s_m$.

1.2 Frequency Component

For $i = 1,2,\ldots,$ let N_i be the random variable representing the number of claims during the i-th period of observation, and let N_i have a Poisson distribution with parameter (mean) Λ. Given m observations $n_1, n_2, \ldots, n_m$, the posterior distribution of Λ is $G(\alpha + m\bar{n}, \beta + m)$ as shown in Section 8.2.2 of Herzog (1999). The parameters α and β determine the prior gamma distribution. The data are summarized by the parameters m and $m\bar{n} = \sum_{i=1}^{m} n_i$. We let $g(\lambda)$ denote the density function of $G(\alpha + m\bar{n}, \beta + m)$.

Then, we are able to write the conditional probability of $N_{m+1} = n_{m+1}$ given $N_1 = n_1, N_2 = n_2, \ldots, N_m = n_m$, as

$$\frac{\int_0^\infty \frac{e^{-\lambda}\lambda^n}{n!}\cdot g(\lambda)d\lambda}{\int_0^\infty g(\lambda)d\lambda} = \frac{\int_0^\infty \frac{e^{-\lambda}\lambda^n}{n!}\cdot e^{-(\beta+m)\lambda}\lambda^{\alpha+m\bar{n}-1}d\lambda}{\int_0^\infty e^{-(\beta+m)\lambda}\lambda^{\alpha+m\bar{n}-1}d\lambda}$$

$$= \frac{\frac{1}{n!}\cdot\int_0^\infty e^{-(\beta+m+1)\lambda}\lambda^{\alpha+m\bar{n}-1+n}d\lambda}{\Gamma(\alpha+m\bar{n})\cdot(\beta+m)^{-(\alpha+m\bar{n})}}$$

$$= \frac{\Gamma(\alpha+m\bar{n}+n)\cdot(\beta+m+1)^{-(\alpha+m\bar{n}+n)}}{n!\Gamma(\alpha+m\bar{n})\cdot(\beta+m)^{-(\alpha+m\bar{n})}}$$

$$= \binom{\alpha+m\bar{n}+n-1}{n}\left(\frac{1}{\beta+m+1}\right)^n\left(\frac{\beta+m}{\beta+m+1}\right)^{\alpha+m\bar{n}},$$

for $n = 1,2,\ldots,$ and is in the form of a negative binomial density function.

1.3 Severity Component

We assume that the amount of each individual claim, X, has an exponential distribution with mean, Δ, and probability density function given by

$$p(x \mid \Delta) = \frac{e^{-x/\Delta}}{\Delta}$$

for $x > 0$ and $\Delta > 0$. The moments of X are $E_X[X \mid \Delta] = \Delta$ and $Var_X(X \mid \Delta) = \Delta^2$. The mean claim amount, Δ, has a conjugate prior distribution whose probability density function, $f(\delta \mid m', y')$, is proportional to $\frac{e^{-y'/\delta}}{\delta^{m'}}$ for $y' > 0, m' > 3,$ and $\delta > 0$ Such a density function is called an *inverse gamma density function*. The insurance process is observed for m periods of observation with n_i claims occurring during period i. The total aggregate claim amount over the m periods of observation is

$$y = \sum_{i=1}^{m\bar{n}} x_i .$$

Then, the posterior density function of Δ, $f(\delta \mid m', y', m\bar{n}, y)$, is proportional to

$$\left(\prod_{i=1}^{m\bar{n}} \frac{e^{-x_i/\delta}}{\delta}\right)\left(\frac{e^{-y'/\delta}}{\delta^{m'}}\right) = \frac{e^{-(y'+y)/\delta}}{\delta^{m'+m\bar{n}}},$$

which is also an inverse gamma density function.

The predictive density function of X, which reflects the uncertainty in the estimation of the parameter values as well as in the random nature of the claim amounts, is given by

$$p(x \mid m', y', m\bar{n}, y) = \int_0^\infty p(x \mid \delta)\cdot f(\delta \mid m', y', m\bar{n}, y)d\delta$$

$$= C\int_0^\infty \left(\frac{e^{-x/\delta}}{\delta}\right)\left(\frac{e^{-(y'+y)/\delta}}{\delta^{m'+m\bar{n}}}\right)d\delta, \qquad (1)$$

where

$$C = \frac{(y'+y)^{m'+m\bar{n}-1}}{\Gamma(m'+m\bar{n}-1)}. \qquad (2)$$

Equation (1) can be rewritten as

$$p(x \mid m', y', m\bar{n}, y) = \int_0^\infty e^{-(x+y'+y)/\delta}\delta^{-m'-m\bar{n}-1}d\delta .$$

Making the change of variable $w = (x+y'+y)/\delta$ and noting that $d\delta = -\frac{x+y'+y}{w^2}dw$, we can rewrite Equation (2) as

$$p(x \mid m', y', m\bar{n}, y) = \left(\frac{(y'+y)^{m'+m\bar{n}-1}}{\Gamma(m'+m\bar{n}-1)}\right)\left(\frac{\int_0^\infty w^{m'+m\bar{n}-1}e^{-w}dw}{(x+y'+y)^{m'+m\bar{n}}}\right)$$

$$= \left(\frac{(y'+y)^{m'+m\bar{n}-1}}{\Gamma(m'+m\bar{n}-1)}\right)\left(\frac{\Gamma(m'+m\bar{n})}{(x+y'+y)^{m'+m\bar{n}}}\right)$$

$$= \frac{(y'+y)^{-1}(m'+m\bar{n}-1)}{\left(1+\frac{x}{y'+y}\right)^{m'+m\bar{n}}},$$

which is a member of the Pareto family of density functions.

We next show how to use pseudo-random numbers and quasi-random numbers to simulate aggregate loss amounts using the predictive distributions for the frequency and severity of insurance claims.

1.4 Simulating Aggregate Losses

1.4.1 Solving the Problem via a Pseudo-Random Number Generator

1.4.1.1 Frequency Component

We assume that the probability of observing n_{m+1} claims during period $m+1$ is given by the negative binomial distribution

$$\Pr[N_{m+1} = n_{m+1}] = \binom{2+n_{m+1}}{n_{m+1}}(.5)^{n_{m+1}}(.5)^3$$

for $n_{m+1} = 0, 1, \ldots$. We employ a pseudo-random number generator in conjunction with the algorithm for the Modified Table-Look-Up Approach to the negative binomial distribution, given in Section 3.2.5.1 of Herzog and Lord (2002), to simulate 10,000 trials of the number of claims. The results are summarized in Table 1.

For the 10,000 trials simulated here we have observed a total of 30,278 claims, which is slightly more than the 3 claims per trial that are expected. (See the discussion in Section 3.2.5 of Herzog and Lord (2002) for more details.)

1.4.1.2 Severity Component

For each of the 30,278 individual claims of the previous section, we need to simulate an individual loss (or claim) amount. We do this by using a pseudo-random number generator to produce uniform random numbers over [0,1), in conjunction with the inversion scheme of Section 3.1.6 of Herzog and Lord (2002) applied to the Pareto probability distribution function given by

$$F(x) = \begin{cases} 0 & x \le 0 \\ 1 - \dfrac{\beta^{\alpha}}{(x+\beta)^{\alpha}} & x > 0 \end{cases},$$

where $\alpha = 20$ and $\beta = 2{,}000{,}000$. In particular, if U is the result of simulating a uniform random variable over [0,1), then the corresponding Pareto random variable is $\frac{\beta}{(1-U)^{1/\alpha}} - \beta$. The results are summarized by the loss severity distribution for which various percentiles are displayed in Table 2.

Table 1: Frequency Component Constructed using Pseudo-Random Number Generator

Number of Claims	Frequency of Occurrence
0	1,234
1	1,852
2	1,910
3	1,548
4	1,110
5	847
6	572
7	356
8	237
9	128
10	93
11	47
12	33
13	14
14	7
15	4
16	6
17	0
18	2
Total	30,278

Table 2: Loss Severity Distribution Constructed using Pseudo-Random Number Generator

Percentile	Point
0	$ 4
10	10,930
25	29,148
50	70,435
75	144,126
90	244,628
100	1,447,454

1.4.1.3 Loss Amounts

Finally, we employ the results of Sections 2.4.1.1 and 2.4.1.2 to produce the distribution of individual loss amounts summarized in Table 3. To illustrate the process, if an individual trial resulted in two claims, then we drew two values from the loss severity distribution.

Table 3: Distribution of Loss Amounts using Pseudo-Random Numbers

Percentile	Point
0	$ 0
10	0
25	71,590
50	229,179
75	471,889
90	753,763
100	2,738,627

1.4.2 Solving the Problem using Quasi-Random Numbers

Because we do not know in advance how many quasi-random numbers we need as input to the algorithm employed to simulate the negative binomial distribution, we can not employ a quasi-Monte Carlo scheme to simulate the number of claims. If we attempted to do so, we would end up with a biased result. However, we can use a quasi-Monte Carlo scheme to simulate the severity portion of the problem, the loss amounts on the 30,278 claims resulting from the first stage of our model.

Our approach is to employ the Neiderreiter sequence

$$\left\{\frac{1}{60{,}556}, \frac{3}{60{,}556}, \ldots, \frac{60{,}555}{60{,}556}\right\},$$

where $60{,}556 = 2 \times 30{,}278$. This gives us an empirical loss severity distribution consisting of 30,278 individual claim amounts in ascending order. The results are summarized in Table 4.

Table 4: Loss Severity Distribution using Quasi-Random Numbers

Percentile	Point
0	$ 2
10	10,563
25	28,974
50	70,526
75	143,540
90	244,011
100	1,468,469

We then employed a pseudo-random number generator to obtain a "random" permutation of the integers from 1 to 30,278, in order to "randomly" re-order (or "shuffle") these loss amounts. The loss amounts are then assigned to an individual trial to produce the distribution of loss amounts summarized in Table 5.

Table 5: Distribution of Loss Amounts using Quasi-random Numbers to Generate the Severity of Loss

Percentile	Point
0	$ 0
10	0
25	67,749
50	227,986
75	467,540
90	753,417
100	2,969,518

Because the quasi-random numbers were "superior" in our previous comparisons, we suspect that the results of Table 5 are "superior" to those of Table 3.

2 MODELING HOME EQUITY CONVERSION MORTGAGES

2.1 Introduction

Many older Americans who own their own homes have most of their wealth in their homes. Some may not otherwise have sufficient wealth to pay for (1) medical bills resulting from sudden, unanticipated medical problems, (2) major repairs to their houses and/or (3) everyday expenses for food, clothing, and so on. Home Equity Conversion Mortgages (HECMs) are designed to allow older people to borrow money by using the equity in their homes as collateral, without being forced to move out of their homes. The amounts borrowed accumulate with interest until the mortgage's due date, at which point the lender is repaid the entire debt.

There are three principal types of HECMs: term, split-term, and tenure. In a term HECM, equal monthly payments are made to the older homeowner for a certain number of months, for example, 180 months or 15 years. At the end of the term, the loan is due and payable. Term HECMs are not popular with older people who fear that they will not be able to repay the loan at the end of the term and be forced out of their homes.

In a split-term HECM, equal monthly payments are made for a certain number of months, but the loan need not be repaid until the older person dies, moves out, or sells his/her house. Finally, in a tenure HECM, equal monthly payments are made and the loan need not be repaid as long as the older person is alive and living in his/her house.

The purpose of this work is to estimate the amount of a level-payment (annuity) of a tenure HECM. We assume an insurance premium structure comprising two components. The first, payable at origination, is equal to 2 percent of the appraised value of the property. The second is an annual insurance fee equal to 0.5 percent of the actual outstanding balance of the loan and is payable monthly. We also assume that the insurer and/or mortgagee has a share of the future appreciation, if any, of the house.

Our HECM model attempts to approximate likely future experience and is flexible in the sense that it can incorporate a wide range of assumptions. Another important feature is that it incorporates the variation associated with the key parameters of the model. Because these parameter values are themselves statistical estimates, such a model more accurately reflects the total variation of the process of interest. This aspect of the model incorporates concepts employed in Herzog and Rubin (1983).

Our results show that viable HECM programs can be constructed by using either a 50/50 shared appreciation scheme (that is, where the mortgagor and insurer and/or mortgagee share future appreciation equally) or one in which the insurer and/or mortgagee receives 100 percent of the nominal appreciation. Of course, the monthly payments are slightly higher in the 100 percent case.

2.2 Assumptions

In this section, we discuss the assumptions of our model.

2.2.1 Appreciation

The annual rate of nominal appreciation of individual houses is a key element of the HECM model. Estimates of the annual rate of nominal appreciation are necessarily imprecise because (1) the rate of appreciation may vary widely from year to year and from neighborhood to neighborhood and (2) the expense of annual appraisals of individual houses makes the attainment of a reliable nationally representative database of U.S. house values impractical.

Our approach to estimating the nominal appreciation of HECM houses is to construct a two-stage stochastic simulation model. In the first stage, we use annual national appreciation data compiled by the National Association of Realtors (NAR)(1989) to simulate the posterior distribution of national appreciation rates. We then use the results of the first-stage model together with some metropolitan NAR data to simulate the posterior distribution of appreciation rates of individual HECM houses.

As shown in the last column of Table 6, the NAR's mean annual rate of increase of the median sales prices of an existing home between December 1981 and December 1988 was 4.26 percent. The corresponding sample variance was 0.000256. The sample autocovariance coefficients of these appreciation rates at lags of one, two, and three years are 0.000110, 0.000029, and 0.00000084, respectively.

Table 6: Annual Appreciation Rates, 1981-1988. Source: National Association of Realtors (1989)

Year	Existing Homes Median Sales Price	Annual Appreciation Rate
1981	$66,600	
1982	$67,800	1.80%
1983	$70,300	3.69%
1984	$72,400	2.99%
1985	$75,500	4.28%
1986	$80,300	6.36%
1987	$85,600	6.60%
1988	$89,100	4.09%
Mean		4.26%

We assume that the first-stage model has a multivariate normal distribution with mean 4.26 percent and variance-covariance matrix equal to 0.0001 times.

$$\begin{bmatrix}
2.56 & 1.10 & 0.29 & 0 & \cdots & & & 0 \\
1.10 & 2.56 & 1.10 & 0.29 & & & & \vdots \\
0.29 & 1.10 & 2.56 & 1.10 & & & & \vdots \\
0 & & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & & & & & & 0 \\
\vdots & & & & 1.10 & 2.56 & 1.10 & 0.29 \\
\vdots & & & & 0.29 & 1.10 & 2.56 & 1.10 \\
0 & \cdots & & & 0 & 0.29 & 1.10 & 2.56
\end{bmatrix}$$

Thus, we assume that the average rate of appreciation over the entire U.S. in year $n+2$ is influenced by the rates of appreciation in years n and $n+1$.

The second-stage model is used to predict the appreciation rates of individual house values. For each year, we use a separate univariate normal distribution whose mean is the corresponding result of the first-stage model and whose standard deviation is 0.08. The value of 0.08 is chosen as a rough measure of the dispersion of the distribution of annual appreciation rates from the first quarter of 1988 to the first quarter of 1989 in the 84 large metropolitan areas of the U. S. considered by Downs (1989). In particular, we note from Appendix B of DiVenti and Herzog (1991) that, based on a mean appreciation rate of 5.21 percent and a standard deviation of 8 percent, we observe one metropolitan area, namely Fort Worth, whose appreciation rate is more than two standard deviations below the mean and five metropolitan areas in California – San Francisco, Orange County, Los Angeles, San Diego, and Riverside – whose appreciation rates are more than two standard deviations above the mean.

The procedures used to generate the random normal deviates required for both stages of the model are described in Chapter 4 of Herzog and Lord (2002).

In addition to 4.26 percent, we also run the model with annual average appreciation rates of 3 percent, 2 percent, and 0 percent. This is because the elderly tend to live in the oldest housing stock, have difficulty keeping their property in good repair, and are unlikely to make home improvements, their property is not likely to appreciate as fast as other property.

2.2.2 Mortality Rates

The basic mortality rates are taken from Wade (1989). Following May and Szymanoski (1989), we assume that all of the mortgagors are single females. This may not be a sufficiently conservative assumption if many married people or other individuals obtain HECMs jointly. Unfortunately, the Social Security Administration cannot provide us with the necessary projected joint mortality rates for married couples. Moreover, our model does not incorporate the likely adverse selection of healthier older people choosing an HECM. Consequently, we recommend that those using this model to price an HECM product make appropriate adjustment for these two factors.

As with the appreciation component, we develop a two-stage stochastic simulation model to predict future mortality experience of HECM mortgagors. In the first stage, we simulate the death rates $q_{65}, q_{70}, \ldots, q_{105}$ using a separate univariate normal model for each death rate. The means of these models are taken from Wade (1989), see Table 7. In particular, we use the value of q_{65+x} projected for calendar year $1990+x$ for $x=0,5,\ldots,40$. We set q_{110} equal to one; that is, we assume that no one survives to age 111.

The standard errors are estimated as follows. We first use the method of least squares to fit a separate linear equation to each of the four sets of 26 values of q_{65+x}, for $x = 0, 5, 10, 15$. The 26 values of the q's are taken from the 1961-1986 U.S. Life Tables for Female Lives, constructed by the National Center for Health Statistics (see Table 8). The standard error of the estimate is used as the estimated standard error of each of these four sets of q's. The remaining standard errors are obtained by fitting a linear equation to the standard errors of the estimates of q_{70}, q_{75}, and q_{80}. The resulting equation is:

$$\text{standard error of } q_{60+5x} = 0.000686x - 0.00074$$

for $x = 5, 6, 7, 8, 9$.

Table 7: Mortality Rate by Year for Annuitants Aged 65 in 1990

q_{65}^{1990}	1.3653%
q_{70}^{1995}	2.0428%
q_{75} 2000	2.8602%
q_{80}^{2005}	4.4065%
q_{85}^{2010}	6.9947%
q_{90}^{2015}	11.5756%
q_{95}^{2020}	17.8137%
q_{100}^{2025}	23.2054%
q_{105}^{2030}	28.7804%

Table 8: U.S. Female Mortality Rates by Age and Calendar Year

Calendar	Age			
Year	65	70	75	80
1961	1.83%	2.84%	4.64%	7.65%
1962	1.84	2.84	4.69	7.73
1963	1.85	2.84	4.71	7.78
1964	1.80	2.73	4.52	7.46
1965	1.79	2.69	4.50	7.44
1966	1.78	2.73	4.52	7.41
1967	1.73	2.66	4.37	7.12
1968	1.78	2.71	4.46	7.29
1969	1.72	2.66	4.32	7.04
1970	1.69	2.64	4.33	6.99
1971	1.62	2.57	4.20	6.75
1972	1.62	2.62	4.24	6.71
1973	1.57	2.53	4.16	6.62
1074	1.51	2.47	3.95	6.30
1975	1.44	2.36	3.77	5.95
1976	1.43	2.30	3.68	5.86
1977	1.42	2.24	3.55	5.65
1978	1.42	2.22	3.48	5.62
1979	1.39	2.15	3.37	5.45
1980	1.44	2.21	3,46	5.61
1981	1.43	2.17	3.39	5.62
1982	1.42	2.13	3.30	5.28
1983	1.40	2.15	3.34	5.39
1984	1.40	2.15	3.33	5.38
1985	1.40	2.15	3.35	5.41
1986	1.40	2.16	3.33	5.34

After the first-stage simulation is run, we obtain the intermediate mortality rates by using a geometric interpolation procedure described on page 272 of Waldman and Gordon (1988). To illustrate this method, we calculate

$$q_{70+x} = q_{70} \cdot \left(\frac{q_{75}}{q_{70}}\right)^{x/5}$$

for $x = 1, 2, 3, 4$.

The second-stage model is a binomial model that simulates the experience of each of the individual insureds. The mortality rates used here are those resulting from the first stage of the model and the interpolation scheme described above. The procedure employed to select the pseudo-random numbers is described in Section 3.2.3 of Herzog and Lord (2002).

Finally, we wonder how the value of the property will be affected if probate problems increase the time it takes the insurer/mortgagee to acquire legal title to the property.

2.2.3 Move-Out Rates

Some mortgagors may move out of their homes and repay their HECM loans because they are in poor health and need to move to a hospital, nursing home, or the home of a friend or relative. Others may move simply because they desire to live in another place. Because their monthly HECM payments terminate in all these instances, we must accurately predict the rate and time at which such moves take place for the population of insureds. Unfortunately, little or no useful data are currently available to construct such estimates. May and Szymanoski (1989) use a rate of 30 percent of the mortality rate at each individual age. We have employed this assumption as well as an assumption of zero. Although we know that zero is too low, it nevertheless does give a measure of the sensitivity of our results to changes in the value of this parameter.

2.2.4 Origination Fees and Other Closing Costs

We assume that at the time the HECM is originated, the mortgagor pays closing costs equal to 1.5 percent of the appraised value of the property. This is intended to cover such costs as the origination fee charged by the lender, the cost of the appraisal of the property, and legal fees. We assume the mortgagor will borrow the closing costs from the lender and incorporate them into the loan.

2.2.5 Transaction Costs

We include estimated transaction costs incurred in selling the house after the older person dies or moves out. Because the real estate sales commission is typically 6 or 7 percent and there are frequently other costs borne by the seller, we assume seller transaction costs of 8 percent of the sale price of the house. If the insurer/mortgagee has to take possession of the property and carry out the preservation costs normally done with a PD (property disposition) property, the transaction costs may be larger than 8 percent. Foster and van Order (1984) used transaction costs of 10 percent of the sale price of the house in their study of defaults on FHA-insured mortgages. We also wonder whether the insurer/mortgagee will be notified promptly after older people die or move out of their homes.

2.2.6 Salaries and Administrative Expenses

We include a component for staff salaries and administrative expenses incurred in running a HECM operation. We set this cost equal to 1 percent of the initial appraised value of the property insured. This rate is comparable to that employed in the principal FHA single-family program.

2.2.7 Interest Rates

We consider three pairs of assumptions for the contract interest rate on the annuity and the discount rate.

Table 9: Interest and Discount Rates

Contract Interest Rate	Discount Rate
8.5%	7.0%
10.0%	8.5%
11.0%	10.0%

2.2.8 House Price

We assume that the HECM is based on an appraised house value of $100,000. This value is selected for mathematical convenience. If the appraised value of the house is less than $100,000, then the amount of the monthly payment should be reduced proportionately. The NAR data shown in the Appendix of DiVenti and Herzog (1991) for the entire U.S. give a median home sales price of $91,600 for the first quarter of calendar year 1989. Hence, even in 1990 a substantial portion of older Americans may have less than $100,000 of equity in their homes. Consequently, their monthly annuity payment would be less than those shown in Table 10.

2.3 Results

We have run each of the first-stage models 10 times and simulated 100 individual HECMs. The mean of the 1,000 simulations is shown in Table 10. These results are sensitive to changes in mean annual appreciation rates, mortality rates, interest rates, and move-out factors. The choice of an appropriate set of assumptions is of course subjective. The insurer/mortgagee naturally must be conservative. By using a move-out factor of 1.0 (to compensate for the high mortality rates resulting from the use of female lives selected from the general population), an annual av-

Table 10: Monthly Annuity Payments Based on a $100,000 House and an Annuitant Age 65 at Purchase

Appreciation Rate	Insurer's Share of Appreciation	Monthly Annuity Payments		
		Contract Interest Rate		
		11.5%	10.0%	8.5%
		Discount Rate		
		10.0%	8.5%	7.0%
Move-out Factor = 1				
4.258%	100%	$335	$379	$433
	50	269	298	334
3.0	100	282	314	352
	50	240	264	292
2.0	100	247	272	302
	50	221	240	264
0.0	100	193	208	226
	50	185	199	215
Move-out Factor = 1.3				
4.258%	100%	$395	388	$439
	50	321	352	388
3.0	100	337	370	410
	50	290	315	344
2.0	100	299	325	365
	50	269	289	314
0.0	100	238	254	273
	50	229	243	261

erage nominal appreciation rate of 2 percent, a contract interest rate of 11.5 percent, and a discount rate of 10.0 percent, we obtain a monthly payment of $221 with a 50/50 shared appreciation HECM and $247 with all the potential appreciation going to the insurer/mortgagee. Hence, HECM instruments may be attractive to some older homeowners. On the other hand, if the insurer decides to decrease the projected mortality rates sharply, increase the standard deviation of the second stage model (say, from 8 percent to 18 or 20 percent), and/or eliminate the shared appreciation feature, then the monthly HECM payment may be so low that no older people will be interested in obtaining one.

3 INTRODUCTION TO VALUE AT RISK

3.1 Introduction

As another example of an application of simulation methods to a practical problem in finance, we consider the topic of *Value at Risk*, a widely-used measure for assessing the risk and/or performance of portfolios of assets and/or liabilities, especially those that include derivatives. At its simplest, Value at Risk (often written VaR) is merely a summary measure of market risk. It provides, in terms of dollars or any other appropriate monetary unit, a number that can be interpreted as an indication of a portfolio's sensitivity to financial market risk. Although it is but one of many possible summary measures, Value at Risk is used extensively (1) in risk management as a measure of capital adequacy and (2) as a measure of the performance of portfolios of assets and/or liabilities.

The widespread acceptance of VaR as a measure of capital adequacy has been accelerated by its endorsement by financial regulators, rating agencies, and certain industry groups.

A basic reference for this topic is *Value at Risk* by Jorian [2000].

3.2 Basic Concepts

The theoretical basis for Value at Risk is found in probability theory while the terminology of VaR comes from the frequentist paradigm of statistics. As with other probabilistic models, the application of VaR entails a number of underlying assumptions. The principal assumption of VaR is that current market conditions will prevail for the immediate future. Then, simply stated, the VaR of the portfolio is the maximum loss anticipated over a given time period for a specified confidence level.

We are now ready to present the basic definition of Value at Risk. We first let W denote the portfolio's value at the beginning of the time period of interest, T denote the length of the time period, and V denote the random variable representing the value of the portfolio at the end of the time period. Then, given a confidence level α, and the assumption(s) on financial market conditions, the Value at Risk, VaR, satisfies the equation

$$\alpha = \Pr[V \geq W - \mathrm{VaR}] = \Pr[W - V \leq \mathrm{VaR}] \tag{3}$$

To be more specific, if α=95% and the length of T is one year, then Equation (3) states that the probability is 95% that the portfolio will lose no more than VaR dollars over the one-year period.

3.3 Estimating Tail Probabilities using Constrained Resampling

The calculation of VaR is equivalent to calculating the percentile of a probability distribution, typically the lower or left-hand tail of the distribution. Frequently, the distribution is multivariate, or for other reasons, may be difficult to handle. For example, in many insurance applications the distribution may have a fat tail. Although simulation methods employing pseudo-random numbers offer a wide variety of methods for probing the distribution's tail, such methods may be inefficient. More efficient solutions can usually be obtained by using constrained sampling in conjunction with low discrepancy sequences.

4 A TEST OF STOCK MARKET EFFICIENCY

The *efficient market hypothesis* contends that fluctuations about the mean market value of a portfolio of common stocks should be capable of being modeled by a sequence of independent, identically distributed random variables. Under such a scenario, common stock prices are said to follow a *random walk*. Various empirical studies call into doubt this hypothesis. Evidence of negative serial correlation of a time series of common stock prices shows that greatly overpriced or under-priced stocks tend to revert toward a mean value. One conjecture – *the rational speculative bubbles hypothesis* – argues that investors realize prices sometimes exceed fundamental values, but believe there is a good probability that the bubble will continue to expand and yield a higher return. The hope of a higher return exactly compensates the investor for the probability of a crash. This model thereby justifies the rationality of holding over-priced stocks. In this application, we first describe a second-order Markov model. We then show how the model can be applied to annual common stock returns in the United States in order to (1) test the efficient market hypothesis and (2) test the rational speculative bubbles hypothesis.

In order to test these two hypotheses, we employ a pair of likelihood ratio tests each of which is asymptotically distributed as a chi-square. Unfortunately, the number of observations is only 39. Consequently, we feel that it is more prudent here to use simulation methods for small samples to estimate the p-value of each test statistic.

The material in this section is based heavily on McQuenn and Thorley [1991]. The rational speculative bubbles hypothesis is from Blanchard and Watson [1982].

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