CHAPTER 7

Advanced Linear Programming

Chapter Guide. This chapter presents the mathematical foundation of linear programming and duality theory. The presentation allows the development of a number of computationally efficient algorithms, including the revised simplex method, bounded variables, and parametric programming. Chapter 20 on the CD presents two additional algorithms that deal with large-scale LPs: decomposition and the Karmarkar interior-point algorithm.

The material in this chapter relies heavily on the use of matrix algebra. Appendix D on the CD provides a review of matrices.

The three topics that should receive special attention in this chapter are the revised simplex method, the bounded-variables algorithm, and parametric programming. The use of matrix manipulations in the revised simplex method allows a better control over machine roundoff error, an ever-present problem in the row operations method of Chapter 3. The bounded variables algorithm is used prominently with the integer programming branch-and-bound algorithm (Chapter 9). Parametric programming adds a dynamic dimension to the LP model that allows the determination of the changes in the optimum solution resulting from making continuous changes in the parameters of the model.

The task of understanding the details of the revised simplex method, bounded variables, decomposition, and parametric programming is improved by summarizing the results of matrix manipulations in the easy-to-read simplex tableau format of Chapter 3. Although matrix manipulations make the algorithms appear different, the theory is exactly the same as in Chapter 3.

This chapter includes 1 real-life application, 8 solved examples, 58 end-of-section problems, and 4 end-of-chapter comprehensive problems. The comprehensive problems are in Appendix E on the CD. The AMPL/Excel/Solver/TORA programs are in folder ch7Files.

Real-Life Application—Optimal Ship Routing and Personnel Assignment for Naval Recruitment in Thailand

Thailand Navy recruits are drafted four times a year. A draftee reports to one of 34 local centers and is then transported by bus to one of four navy branch bases. From there, recruits are transported to the main naval base by ship. The docking facilities at the branch bases may restrict the type of ship that can visit each base. Branch bases have limited capacities but, as a whole, the four bases have sufficient capacity to accommodate all the draftees. During the summer of 1983, a total of 2929 draftees were transported from the drafting centers to the four branch bases and eventually to the main base. The problem deals with determining the optimal schedule for transporting the draftees, first from the drafting centers to the branch bases and then from the branch bases to the main base. The study uses a combination of linear and integer programming. The details are given in Case 5, Chapter 24 on the CD.

7.1 SIMPLEX METHOD FUNDAMENTALS

In linear programming, the feasible solution space is said to form a **convex set** if the line segment joining any two *distinct* feasible points also falls in the set. An **extreme point** of the convex set is a feasible point that cannot lie on a line segment joining any two *distinct* feasible points in the set. Actually, extreme points are the same as corner point, the more apt name used in Chapters 2, 3, and 4.

Figure 7.1 illustrates two sets. Set (a), which is typical of the solution space of a linear program, is convex (with six extreme points), whereas set (b) is nonconvex.

In the graphical LP solution given in Section 2.3, we demonstrated that the optimum solution can always be associated with a feasible extreme (corner) point of the solution space. This result makes sense intuitively, because in the LP solution space every feasible point can be determined as a function of its feasible extreme points. For example, in convex set (a) of Figure 7.1, a feasible point $\mathbf{X}$ can be expressed as a **convex combination** of its extreme points $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_4$, $\mathbf{X}_5$, and $\mathbf{X}_6$ using

$$\mathbf{X} = \alpha_1\mathbf{X}_1 + \alpha_2\mathbf{X}_2 + \alpha_3\mathbf{X}_3 + \alpha_4\mathbf{X}_4 + \alpha_5\mathbf{X}_5 + \alpha_6\mathbf{X}_6$$

where

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 1$$

$$\alpha_i \geq 0, i = 1, 2, \ldots, 6$$

This observation shows that extreme points provide all that is needed to define the solution space completely.

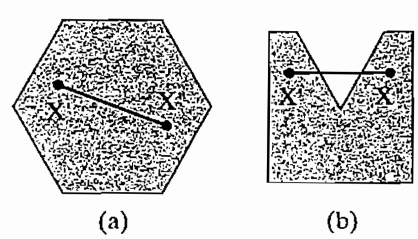


FIGURE 7.1
Examples of a convex and a nonconvex set

Example 7.1-1

Show that the following set is convex:

$$C = \{(x_1, x_2) | x_1 \le 2, x_2 \le 3, x_1 \ge 0, x_2 \ge 0\}$$

Let $\mathbf{X}_1 = \{x_1', x_2'\}$ and $\mathbf{X}_2 = \{x_1'', x_2''\}$ be any two distinct points in C. If C is convex, then $\mathbf{X} = (x_1, x_2) = \alpha_1\mathbf{X}_1 + \alpha_2\mathbf{X}_2$, $\alpha_1 + \alpha_2 = 1, \alpha_1, \alpha_2 \ge 0$, must also be in C. To show that this is true, we need to show that all the constraints of C are satisfied by the line segment $\mathbf{X}$; that is,

$$x_1 = \alpha_1 x_1' + \alpha_2 x_1'' \le \alpha_1(2) + \alpha_2(2) = 2$$
$$x_2 = \alpha_1 x_2' + \alpha_2 x_2'' \le \alpha_1(3) + \alpha_2(3) = 3$$

Thus, $x_1 \le 2$ and $x_2 \le 3$. Additionally, the nonnegativity conditions are satisfied because α_1 and α_2 are nonnegative.

PROBLEM SET 7.1A

1. Show that the set $Q = \{x_1, x_2 | x_1 + x_2 \le 1, x_1 \ge 0, x_2 \ge 0\}$ is convex. Is the nonnegativity condition essential for the proof?

*2. Show that the set $Q = \{x_1, x_2 | x_1 \ge 1 \text{ or } x_2 \ge 2\}$ is not convex.

3. Determine graphically the extreme points of the following convex set:

$$Q = \{x_1, x_2 | x_1 + x_2 \le 2, x_1 \ge 0, x_2 \ge 0\}$$

Show that the entire feasible solution space can be determined as a convex combination of its extreme points. Hence conclude that any convex (bounded) solution space is totally defined once its extreme points are known.

4. In the solution space in Figure 7.2 (drawn to scale), express the interior point (3, 1) as a convex combination of the extreme points A, B, C, and D with each extreme point carrying a strictly positive weight.

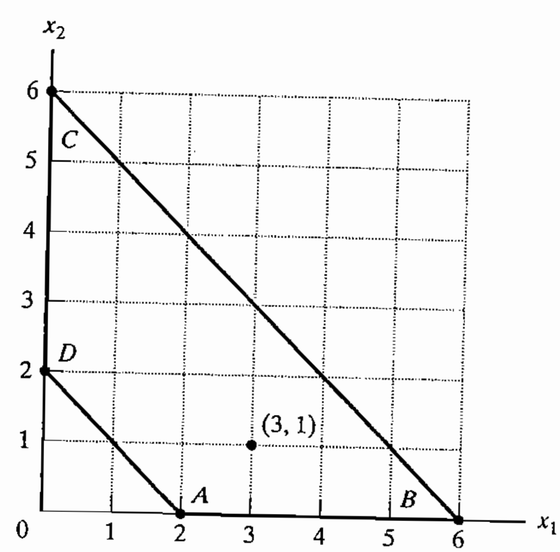


FIGURE 7.2

Solution space for Problem 4, Set 7.1a

7.1.1 From Extreme Points to Basic Solutions

It is convenient to express the general LP problem in equation form (see Section 3.1) using matrix notation. Define $\mathbf{X}$ as an n-vector representing the variables, $\mathbf{A}$ as an $(m \times n)$-matrix representing the constraint coefficients, $\mathbf{b}$ as a column vector representing the right-hand side, and $\mathbf{C}$ as an n-vector representing the objective-function coefficients. The LP is then written as

$$\text{Maximize or minimize } z = \mathbf{CX}$$

subject to

$$\mathbf{AX} = \mathbf{b}$$
$$\mathbf{X} \geq \mathbf{0}$$

Using the format of Chapter 3 (see also Figure 4.1), the rightmost m columns of $\mathbf{A}$ always can be made to represent the identity matrix $\mathbf{I}$ through proper arrangements of the slack/artificial variables associated with the starting basic solution.

A **basic solution** of $\mathbf{AX} = \mathbf{b}$ is determined by setting $n - m$ variables equal to zero, and then solving the resulting m equations in the remaining m unknowns, *provided that the resulting solution is unique*. Given this definition, the theory of linear programming establishes the following result between the geometric definition of extreme points and the algebraic definition of basic solutions:

$$\text{Extreme points of } \{\mathbf{X} | \mathbf{AX} = \mathbf{b}\} \Leftrightarrow \text{Basic solutions of } \mathbf{AX} = \mathbf{b}$$

The relationship means that the extreme points of the LP solution space are totally defined by the basic solutions of the system $\mathbf{AX} = \mathbf{b}$, and vice versa. Thus, we conclude that the basic solutions of $\mathbf{AX} = \mathbf{b}$ contain all the information we need to determine the optimum solution of the LP problem. Furthermore, if we impose the nonnegativity restriction, $\mathbf{X} \geq \mathbf{0}$, the search for the optimum solution is confined to the *feasible* basic solutions only.

To formalize the definition of a basic solution, the system $\mathbf{AX} = \mathbf{b}$ can be expressed in vector form as follows:

$$\sum_{j=1}^{n} \mathbf{P}_j x_j = \mathbf{b}$$

The vector $\mathbf{P}_j$ is the jth column of $\mathbf{A}$. A subset of m vectors is said to form a **basis, B,** if, and only if, the selected m vectors are **linearly independent**. In this case, the matrix $\mathbf{B}$ is **nonsingular**. If $\mathbf{X}_B$ is the set of m variables associated with the vectors of nonsingular $\mathbf{B}$, then $\mathbf{X}_B$ must be a basic solution. In this case, we have

$$\mathbf{BX}_B = \mathbf{b}$$

Given the inverse $\mathbf{B}^{-1}$ of $\mathbf{B}$, we then get the corresponding basic solution as

$$\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b}$$

If $\mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, then $\mathbf{X}_B$ is feasible. The definition assumes that the remaining $n - m$ variables are **nonbasic** at zero level.

The previous result shows that in a system of m equations and n unknowns, the maximum number of (feasible and infeasible) basic solutions is given by

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

Example 7.1-2

Determine and classify (as feasible and infeasible) all the basic solutions of the following system of equations.

$$\begin{pmatrix} 1 & 3 & -1 \\ 2 & -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

The following table summarizes the results. The inverse of **B** is determined by using one of the methods in Section D.2.7 on the CD.

B	$\mathbf{BX}_B = \mathbf{b}$	Solution	Status
$(\mathbf{P}_1, \mathbf{P}_2)$	$\begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$	$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{7}{4} \\ \frac{3}{4} \end{pmatrix}$	Feasible
$(\mathbf{P}_1, \mathbf{P}_3)$	(Not a basis)	—	—
$(\mathbf{P}_2, \mathbf{P}_3)$	$\begin{pmatrix} 3 & -1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$	$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{3}{8} \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ -\frac{7}{4} \end{pmatrix}$	Infeasible

We can also investigate the problem by expressing it in vector form as follows:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}x_1 + \begin{pmatrix} 3 \\ -2 \end{pmatrix}x_2 + \begin{pmatrix} -1 \\ -2 \end{pmatrix}x_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Each of $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{b}$ is a two-dimensional vector, which can be represented generically as $(a_1, a_2)^T$. Figure 7.3 graphs these vectors on the (a_1, a_2)-plane. For example, for $\mathbf{b} = (4, 2)^T$, $a_1 = 4$ and $a_2 = 2$.

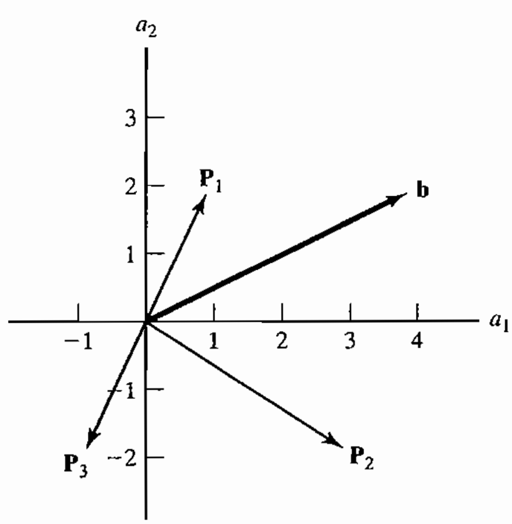


FIGURE 7.3

Vector representation of LP solution space

Because we are dealing with two equations ($m = 2$), a basis must include exactly two vectors, selected from among $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. From Figure 7.3, the matrices $(\mathbf{P}_1, \mathbf{P}_2)$ and $(\mathbf{P}_2, \mathbf{P}_3)$ form bases because their associated vectors are independent. In the matrix $(\mathbf{P}_1, \mathbf{P}_3)$ the two vectors are dependent, and hence do not constitute a basis.

Algebraically, a (square) matrix forms a basis if its determinant is not zero (see Section D.2.5). The following computations show that the combinations $(\mathbf{P}_1, \mathbf{P}_2)$ and $(\mathbf{P}_2, \mathbf{P}_3)$ are bases, and the combination $(\mathbf{P}_1, \mathbf{P}_3)$ is not.

$$\det(\mathbf{P}_1, \mathbf{P}_2) = \det\begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix} = (1 \times -2) - (2 \times 3) = -8 \neq 0$$

$$\det(\mathbf{P}_2, \mathbf{P}_3) = \det\begin{pmatrix} 3 & -1 \\ -2 & -2 \end{pmatrix} = (3 \times -2) - (-2 \times -1) = -8 \neq 0$$

$$\det(\mathbf{P}_1, \mathbf{P}_3) = \det\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = (1 \times -2) - (2 \times -1) = 0$$

We can take advantage of the vector representation of the problem to discuss how the starting solution of the simplex method is determined. From the vector representation in Figure 7.3, the basis $\mathbf{B} = (\mathbf{P}_1, \mathbf{P}_2)$ can be used to start the simplex iterations, because it produces the basic feasible solution $\mathbf{X}_B = (x_1, x_2)^T$. However, in the absence of the vector representation, the only available course of action is to try all possible bases (3 in this example, as shown above). The difficulty with using trial and error is that it is not suitable for automatic computations. In a typical LP with thousands of variables and constraints where the use of the computer is a must, trial and error is not a practical option because of its tremendous computational overhead. To alleviate this problem, the simplex method always uses an identity matrix, $\mathbf{B} = \mathbf{I}$, to start the iterations. Why does a starting $\mathbf{B} = \mathbf{I}$ offer an advantage? The answer is that it will always provide a *feasible* starting basic solution (provided that the right-hand side vector of the equations is nonnegative). You can see this result in Figure 7.3 by graphing the vectors of $\mathbf{B} = \mathbf{I}$ and noting that they coincide with the horizontal and vertical axes, thus always guaranteeing a starting basic feasible solution.

The basis $\mathbf{B} = \mathbf{I}$ automatically forms part of the LP equations if all the original constraints are $\leq$. In other cases, we simply add the unit vectors where needed. This is what the artificial variables accomplish (Section 3.4). We then penalize these variables in the objective function to force them to zero level in the final solution.

PROBLEM SET 7.1B

1. In the following sets of equations, (a) and (b) have unique (basic) solutions, (c) has an infinity of solutions, and (d) has no solution. Show how these results can be verified using graphical *vector* representation. From this exercise, state the general conditions for vector dependence-independence that lead to unique solution, infinity of solutions, and no solution.

(a) $x_1 + 3x_2 = 2$
$3x_1 + x_2 = 3$

(b) $2x_1 + 3x_2 = 1$
$2x_1 - x_2 = 2$

(c) $2x_1 + 6x_2 = 4$
$x_1 + 3x_2 = 2$

(d) $2x_1 - 4x_2 = 2$
$-x_1 + 2x_2 = 1$

2. Use vectors to determine graphically the type of solution for each of the sets of equations below: unique solution, an infinity of solutions, or no solution. For the cases of unique solutions, indicate from the vector representation (and without solving the equations algebraically) whether the values of the x_1 and x_2 are positive, zero, or negative.

(a) $\begin{pmatrix} 5 & 4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *__(b)__ $\begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$ *__(d)__ $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

(e) $\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *__(f)__ $\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

3. Consider the following system of equations:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} x_3 + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} x_4 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

Determine if any of the following combinations forms a basis.

*__(a)__ $(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$

(b) $(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_4)$

(c) $(\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4)$

*__(d)__ $(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4)$

4. True or False?

(a) The system $\mathbf{BX} = \mathbf{b}$ has a unique solution if $\mathbf{B}$ is nonsingular.

(b) The system $\mathbf{BX} = \mathbf{b}$ has no solution if $\mathbf{B}$ is singular and $\mathbf{b}$ is independent of $\mathbf{B}$.

(c) The system $\mathbf{BX} = \mathbf{b}$ has an infinity of solutions if $\mathbf{B}$ is singular and $\mathbf{b}$ is dependent.

7.1.2 Generalized Simplex Tableau in Matrix Form

In this section, we use matrices to develop the general simplex tableau. This representation will be the basis for subsequent developments in the chapter.

Consider the LP in equation form:

$$\text{Maximize } z = \mathbf{CX}, \text{ subject to } \mathbf{AX} = \mathbf{b}, \mathbf{X} \geq \mathbf{0}$$

The problem can be written equivalently as

$$\begin{pmatrix} 1 & -\mathbf{C} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}\begin{pmatrix} z \\ \mathbf{X} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}$$

Suppose that $\mathbf{B}$ is a feasible basis of the system $\mathbf{AX} = \mathbf{b}, \mathbf{X} \geq \mathbf{0}$, and let $\mathbf{X}$ be the corresponding vector of basic variables with $\mathbf{C}_B$ as its associated objective vector. The corresponding solution may then be computed as follows (the method for inverting partitioned matrices is given in Section D.2.7):

$$\begin{pmatrix} z \\ \mathbf{X}_B \end{pmatrix} = \begin{pmatrix} 1 & -\mathbf{C}_B \\ \mathbf{0} & \mathbf{B} \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{C}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{pmatrix}$$

The general simplex tableau in matrix form can be derived from the original standard equations as follows:

$$\begin{pmatrix} 1 & \mathbf{C}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{pmatrix}\begin{pmatrix} 1 & -\mathbf{C} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}\begin{pmatrix} z \\ \mathbf{X} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{C}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}$$

Matrix manipulations yield the following equations:

$$\begin{pmatrix} 1 & \mathbf{C}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{C} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} \end{pmatrix}\begin{pmatrix} z \\ \mathbf{X} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{pmatrix}$$

Given $\mathbf{P}_j$ is the jth vector of $\mathbf{A}$, the simplex tableau column associated with variable x_j can be represented as:

Basic	x_j	Solution
z	$\mathbf{C}_B\mathbf{B}^{-1}\mathbf{P}_j - c_j$	$\mathbf{C}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{X}_B$	$\mathbf{B}^{-1}\mathbf{P}_j$	$\mathbf{B}^{-1}\mathbf{b}$

In fact, the tableau above is the same as the one we presented in Chapter 3 (see Problem 5 of Set 7.1c) and that of the primal-dual computations in Section 4.2.4. An important property of this table is that the inverse, $\mathbf{B}^{-1}$, is the only element that changes from one tableau to the next, and that the *entire* tableau can be generated once $\mathbf{B}^{-1}$ is known. This point is important, because the computational roundoff error in any tableau can controlled by controlling the accuracy of $\mathbf{B}^{-1}$. This result is the basis for the development of the revised simplex method in Section 7.2.

Example 7.1-3

Consider the following LP:

$$\text{Maximize } z = x_1 + 4x_2 + 7x_3 + 5x_4$$

subject to

$$2x_1 + x_2 + 2x_3 + 4x_4 = 10$$

$$3x_1 - x_2 - 2x_3 + 6x_4 = 5$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Generate the simplex tableau associated with the basis $\mathbf{B} = (\mathbf{P}_1, \mathbf{P}_2)$.

Given $\mathbf{B} = (\mathbf{P}_1, \mathbf{P}_2)$, then $\mathbf{X}_B = (x_1, x_2)^T$ and $\mathbf{C}_B = (1, 4)$. Thus,

$$\mathbf{B}^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}$$

We then get

$$\mathbf{X}_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}\begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

To compute the constraint columns in the body of the tableau, we have

$$\mathbf{B}^{-1}(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4) = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}\begin{pmatrix} 2 & 1 & 2 & 4 \\ 3 & -1 & -2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

Next, we compute the objective row as follows:

$$\mathbf{C}_B(\mathbf{B}^{-1}(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_2, \mathbf{P}_4)) - \mathbf{C} = (1, 4)\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} - (1, 4, 7, 5) = (0, 0, 1, -3)$$

Finally, we compute the value of the objective function as follows:

$$z = \mathbf{C}_B\mathbf{B}^{-1}\mathbf{b} = \mathbf{C}_B\mathbf{X}_B = (1, 4)\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 19$$

Thus, the entire tableau can be summarized as shown below.

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	1	-3	19
x_1	1	0	0	2	3
x_2	0	1	2	0	4

The main conclusion from this example is that once the inverse, $\mathbf{B}^{-1}$, is known, the entire simplex tableau can be generated from $\mathbf{B}^{-1}$ and the *original* data of the problem.

PROBLEM SET 7.1C

***1.** In Example 7.1-3, consider $\mathbf{B} = (\mathbf{P}_3, \mathbf{P}_4)$. Show that the corresponding basic solution is feasible, then generate the corresponding simplex tableau.

2. Consider the following LP:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 10 \\ 2x_1 - 2x_2 - x_3 \quad\quad &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Check if each of the following matrices forms a (feasible or infeasible) basis: $(\mathbf{P}_1, \mathbf{P}_2)$, $(\mathbf{P}_2, \mathbf{P}_3)$, $(\mathbf{P}_3, \mathbf{P}_4)$.

3. In the following LP, compute the entire simplex tableau associated with $\mathbf{X}_B = (x_1, x_2, x_5)^T$.

$$\text{Minimize } z = 2x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 - x_3 \qquad\qquad &= 3 \\ 4x_1 + 3x_2 \qquad - x_4 \qquad &= 6 \\ x_1 + 2x_2 \qquad\qquad + x_5 &= 3 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

***4.** The following is an optimal LP tableau:

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	0	3	2	?
x_3	0	0	1	1	-1	2
x_2	0	1	0	1	0	6
x_1	1	0	0	-1	1	2

The variables x_3, x_4, and x_5 are slacks in the original problem. Use matrix manipulations to reconstruct the original LP, and then compute the optimum value.

5. In the generalized simplex tableau, suppose that the $\mathbf{X} = (\mathbf{X}_\mathrm{I}, \mathbf{X}_\mathrm{II})^T$, where $\mathbf{X}_\mathrm{II}$ corresponds to a typical *starting* basic solution (consisting of slack and/or artificial variables) with $\mathbf{B} = \mathbf{I}$, and let $\mathbf{C} = (\mathbf{C}_\mathrm{I}, \mathbf{C}_\mathrm{II})$ and $\mathbf{A} = (\mathbf{D}, \mathbf{I})$ be the corresponding partitions of $\mathbf{C}$ and $\mathbf{A}$, respectively. Show that the matrix form of the simplex tableau reduces to the following form, which is exactly the form used in Chapter 3.

Basic	$\mathbf{X}_\mathrm{I}$	$\mathbf{X}_\mathrm{II}$	Solution
z	$\mathbf{C}_B\mathbf{B}^{-1}\mathbf{D} - \mathbf{C}_\mathrm{I}$	$\mathbf{C}_B\mathbf{B}^{-1} - \mathbf{C}_\mathrm{II}$	$\mathbf{C}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{X}_B$	$\mathbf{B}^{-1}\mathbf{D}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

7.2 REVISED SIMPLEX METHOD

Section 7.1.1 shows that the optimum solution of a linear program is always associated with a basic (feasible) solution. The simplex method search for the optimum starts by selecting a feasible basis, $\mathbf{B}$, and then moving to another basis, $\mathbf{B}_{\text{next}}$, that yields a better (or, at least, no worse) value of the objective function. Continuing in this manner, the optimum basis is eventually reached.

The iterative steps of the *revised* simplex method are exactly the same as in the *tableau* simplex method presented in Chapter 3. The main difference is that the computations in the revised method are based on matrix manipulations rather than on row operations. The use of matrix algebra reduces the adverse effect of machine roundoff error by controlling the accuracy of computing $\mathbf{B}^{-1}$. This result follows because, as Section 7.1.2 shows, the entire simplex tableau can be computed from the *original* data and the current $\mathbf{B}^{-1}$. In the tableau simplex method of Chapter 3, each tableau is generated from the immediately preceding one, which tends to worsen the problem of rounoff error.

7.2.1 Development of the Optimality and Feasibility Conditions

The general LP problem can be written as follows:

$$\text{Maximize or minimize } z = \sum_{j=1}^{n} c_j x_j \text{ subject to } \sum_{j=1}^{n} \mathbf{P}_j x_j = \mathbf{b},\ x_j \geq 0,\ j = 1, 2, \ldots, n$$

For a given basic vector $\mathbf{X}_B$ and its corresponding basis $\mathbf{B}$ and objective vector $\mathbf{C}_B$, the general simplex tableau developed in Section 7.1.2 shows that any simplex iteration can be represented by the following equations:

$$z + \sum_{j=1}^{n} (z_j - c_j) x_j = \mathbf{C}_B \mathbf{B}^{-1} \mathbf{b}$$

$$(\mathbf{X}_B)_i + \sum_{j=1}^{n} (\mathbf{B}^{-1} \mathbf{P}_j)_i \, x_j = (\mathbf{B}^{-1} \mathbf{b})_i$$

$z_j - c_j$, the reduced cost of x_j (see Section 4.3.2), is defined as

$$z_j - c_j = \mathbf{C}_B \mathbf{B}^{-1} \mathbf{P}_j - c_j$$

The notation $(\mathbf{V})_i$ is used to represent the ith element of the vector $\mathbf{V}$.

Optimality Condition. From the z-equation given above, an increase in nonbasic x_j above its current zero value will improve the value of z relative to its current value $(= \mathbf{C}_B \mathbf{B}^{-1} \mathbf{b})$ only if its $z_j - c_j$ is strictly negative in the case of maximization and strictly positive in the case of minimization. Otherwise, x_j cannot improve the solution and must remain nonbasic at zero level. Though any nonbasic variable satisfying the given condition can be chosen to improve the solution, the simplex method uses a rule of thumb that calls for selecting the **entering variable** as the one with the *most* negative (*most* positive) $z_j - c_j$ in case of maximization (minimization).

Feasibility Condition. The determination of the **leaving vector** is based on examining the constraint equation associated with the ith *basic* variable. Specifically, we have

$$(\mathbf{X}_B)_i + \sum_{j=1}^{n} (\mathbf{B}^{-1} \mathbf{P}_j)_i \, x_j = (\mathbf{B}^{-1} \mathbf{b})_i$$

When the vector $\mathbf{P}_j$ is selected by the optimality condition to enter the basis, its associated variable x_j will increase above zero level. At the same time, all the remaining nonbasic variables remain at zero level. Thus, the ith constraint equation reduces to

$$(\mathbf{X}_B)_i = (\mathbf{B}^{-1} \mathbf{b})_i - (\mathbf{B}^{-1} \mathbf{P}_j)_i \, x_j$$

The equation shows that if $(\mathbf{B}^{-1} \mathbf{P}_j)_i > 0$, an increase in x_j can cause $(\mathbf{X}_B)_i$ to become negative, which violates the nonnegativity condition, $(\mathbf{X}_B)_i \geq \mathbf{0}$ for all i. Thus, we have

$$(\mathbf{B}^{-1} \mathbf{b})_i - (\mathbf{B}^{-1} \mathbf{P}_j)_i \, x_j \geq 0, \quad \text{for all } i$$

This condition yields the maximum value of the entering variable x_j as

$$x_j = \min_i \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{P}_j)_i} \middle| (\mathbf{B}^{-1}\mathbf{P}_j)_i > 0 \right\}$$

The basic variable responsible for producing the minimum ratio leaves the basic solution to become nonbasic at zero level.

PROBLEM SET 7.2A

*1. Consider the following LP:

$$\text{Maximize } z = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

subject to

$$\mathbf{P}_1x_1 + \mathbf{P}_2x_2 + \mathbf{P}_3x_3 + \mathbf{P}_4x_4 = \mathbf{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

The vectors $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ are shown in Figure 7.4. Assume that the basis $\mathbf{B}$ of the current iteration is comprised of $\mathbf{P}_1$ and $\mathbf{P}_2$.

(a) If the vector $\mathbf{P}_3$ enters the basis, which of the current two basic vectors must leave in order for the resulting basic solution to be feasible?

(b) Can the vector $\mathbf{P}_4$ be part of a feasible basis?

*2. Prove that, in any simplex iteration, $z_j - c_j = 0$ for all the associated *basic* variables.

3. Prove that if $z_j - c_j > 0$ (< 0) for all the nonbasic variables x_j of a maximization (minimization) LP problem, then the optimum is unique. Else, if $z_j - c_j$ equals zero for a nonbasic x_j, then the problem has an alternative optimum solution.

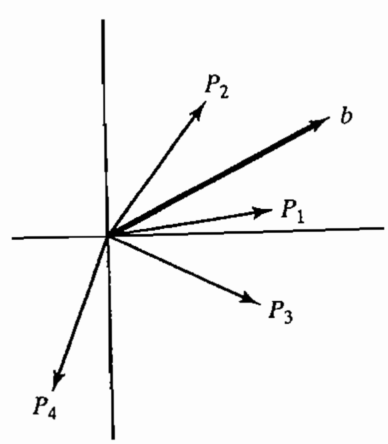
4. In an all-slack starting basic solution, show using the matrix form of the tableau that the mechanical procedure used in Section 3.3 in which the objective equation is set as

$$z - \sum_{j=1}^{n} c_jx_j = 0$$

automatically computes the proper $z_j - c_j$ for all the variables in the starting tableau.

5. Using the matrix form of the simplex tableau, show that in an all-artificial starting basic solution, the procedure employed in Section 3.4.1 that calls for substituting out

FIGURE 7.4
Vector representation of Problem 1, Set 7.2a



7.2.

the artificial variables in the objective function (using the constraint equations) actually computes the proper $z_j - c_j$ for all the variables in the starting tableau.

6. Consider an LP in which the variable x_k is unrestricted in sign. Prove that by substituting $x_k = x_k^- - x_k^+$, where x_k^- and x_k^+ are nonnegative, it is impossible that the two variables will replace one another in an alternative optimum solution.

*7. Given the general LP in equation form with m equations and n unknowns, determine the maximum number of *adjacent* extreme points that can be reached from a nondegenerate extreme point (all basic variable are >0) of the solution space.

8. In applying the feasibility condition of the simplex method, suppose that $x_r = 0$ is a basic variable and that x_j is the entering variable with $(\mathbf{B}^{-1}\mathbf{P}_j)_r \neq 0$. Prove that the resulting basic solution remains feasible even if $(\mathbf{B}^{-1}\mathbf{P}_j)_r$ is negative.

9. In the implementation of the feasibility condition of the simplex method, what are the conditions for encountering a degenerate solution (at least one basic variable $= 0$) for the first time? For continuing to obtain a degenerate solution in the next iteration? For removing degeneracy in the next iteration? Explain the answers mathematically.

*10. What are the relationships between extreme points and basic solutions under degeneracy and nondegeneracy? What is the maximum number of iterations that can be performed at a given extreme point assuming no cycling?

*11. Consider the LP, maximize $z = \mathbf{CX}$ subject to $\mathbf{AX} \leq \mathbf{b}, \mathbf{X} \geq \mathbf{0}$, where $\mathbf{b} \geq \mathbf{0}$. Suppose that the entering vector $\mathbf{P}_j$ is such that at least one element of $\mathbf{B}^{-1}\mathbf{P}_j$ is positive.
    (a) If $\mathbf{P}_j$ is replaced with $\alpha\mathbf{P}_j$, where α is a positive scalar, and provided x_j remains the entering variable, find the relationship between the values of x_j corresponding to $\mathbf{P}_j$ and $\alpha\mathbf{P}_j$.
    (b) Answer Part (a) if, additionally, $\mathbf{b}$ is replaced with $\beta\mathbf{b}$, where β is a positive scalar.

12. Consider the LP

$$\text{Maximize } z = \mathbf{CX} \text{ subject to } \mathbf{AX} \leq \mathbf{b}, \mathbf{X} \geq \mathbf{0}, \text{ where } \mathbf{b} \geq \mathbf{0}$$

After obtaining the optimum solution, it is suggested that a nonbasic variable x_j can be made basic (profitable) by reducing the (resource) requirements per unit of x_j for the different resources to $\frac{1}{\alpha}$ of their original values, $\alpha > 1$. Since the requirements per unit are reduced, it is expected that the profit per unit of x_j will also be reduced to $\frac{1}{\alpha}$ of its original value. Will these changes make x_j a profitable variable? Explain mathematically.

13. Consider the LP

$$\text{Maximize } z = \mathbf{CX} \text{ subject to } (\mathbf{A}, \mathbf{I})\mathbf{X} = \mathbf{b}, \mathbf{X} \geq \mathbf{0}$$

Define $\mathbf{X}_B$ as the current basic vector with $\mathbf{B}$ as its associated basis and $\mathbf{C}_B$ as its vector of objective coefficients. Show that if $\mathbf{C}_B$ is replaced with the new coefficients $\mathbf{D}_B$, the values of $z_j - c_j$ for the basic vector $\mathbf{X}_B$ will remain equal to zero. What is the significance of this result?

7.2.2 Revised Simplex Algorithm

Having developed the optimality and feasibility conditions in Section 7.2.1, we now present the computational steps of the revised simplex method.

Step 0. Construct a starting basic feasible solution and let $\mathbf{B}$ and $\mathbf{C}_B$ be its associated basis and objective coefficients vector, respectively.

Step 1. Compute the inverse $\mathbf{B}^{-1}$ by using an appropriate inversion method.[1]

Step 2. For each *nonbasic* variable x_j, compute

$$z_j - c_j = \mathbf{C}_B\mathbf{B}^{-1}\mathbf{P}_j - c_j$$

If $z_j - c_j \geq 0$ in maximization (≤ 0 in minimization) for all nonbasic x_j, stop; the optimal solution is given by

$$\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b},\ z = \mathbf{C}_B\mathbf{X}_B$$

Else, apply the optimality condition and determine the *entering* variable x_j as the nonbasic variable with the most negative (positive) $z_j - c_j$ in case of mazimization (minimization).

Step 3. Compute $\mathbf{B}^{-1}\mathbf{P}_j$. If all the elements of $\mathbf{B}^{-1}\mathbf{P}_j$ are negative or zero, stop; the problem has no bounded solution. Else, compute $\mathbf{B}^{-1}\mathbf{b}$. Then for all the *strictly positive* elements of $\mathbf{B}^{-1}\mathbf{P}_j$, determine the ratios defined by the feasibility condition. The basic variable x_i associated with the smallest ratio is the *leaving* variable.

Step 4. From the current basis $\mathbf{B}$, form a new basis by replacing the leaving vector $\mathbf{P}_i$ with the entering vector $\mathbf{P}_j$, Go to step 1 to start a new iteration.

Example 7.2-1

The Reddy Mikks model (Section 2.1) is solved by the revised simplex algorithm. The same model was solved by the tableau method in Section 3.3.2. A comparison between the two methods will show that they are one and the same.

The equation form of the Reddy Mikks model can be expressed in matrix form as

$$\text{maximize } z = (5, 4, 0, 0, 0, 0)(x_1, x_2, x_3, x_4, x_5, x_6)^T$$

subject to

$$\begin{pmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1, x_2, \ldots, x_6 \geq 0$$

We use the notation $\mathbf{C} = (c_1, c_2, \ldots, c_6)$ to represent the objective-function coefficients and $(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_6)$ to represent the columns vectors of the constraint equations. The right-hand side of the constraints gives the vector $\mathbf{b}$.

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[1]In most LP presentations, including the first six editions of this book, the *product form* method for inverting a basis (see Section D.2.7) is integrated into the revised simplex algorithm because the *product form* lends itself readily to the revised simplex computations, where successive bases differ in exactly one column. This detail is removed from this presentation because it makes the algorithm appear more complex than it really is. Moreover, the *product form* is rarely used in the development of LP codes because it is not designed for automatic computations, where machine round-off error can be a serious issue. Normally, some advanced numeric analysis method, such as the *LU decomposition* method, is used to obtain the inverse. (Incidentally, TORA matrix inversion is based on LU decomposition.)

In the computations below, we will give the algebraic formula for each step and its final numeric answer without detailing the arithmetic operations. You will find it instructive to fill in the gaps in each step.

Iteration 0

$$\mathbf{X}_{B_0} = (x_3, x_4, x_5, x_6), \mathbf{C}_{B_0} = (0, 0, 0, 0)$$
$$\mathbf{B}_0 = (\mathbf{P}_3, \mathbf{P}_4, \mathbf{P}_5, \mathbf{P}_6) = \mathbf{I}, \mathbf{B}_0^{-1} = \mathbf{I}$$

Thus,

$$\mathbf{X}_{B_0} = \mathbf{B}_0^{-1}\mathbf{b} = (24, 6, 1, 2)^T, z = \mathbf{C}_{B_0}\mathbf{X}_{B_0} = 0$$

Optimality computations:

$$\mathbf{C}_{B_0}\mathbf{B}_0^{-1} = (0, 0, 0, 0)$$
$$\{z_j - c_j\}_{j=1,2} = \mathbf{C}_{B_0}\mathbf{B}_0^{-1}(\mathbf{P}_1, \mathbf{P}_2) - (c_1, c_2) = (-5, -4)$$

Thus, $\mathbf{P}_1$ is the entering vector.

Feasibility computations:

$$\mathbf{X}_{B_0} = (x_3, x_4, x_5, x_6)^T = (24, 6, 1, 2)^T$$
$$\mathbf{B}_0^{-1}\mathbf{P}_1 = (6, 1, -1, 0)^T$$

Hence,

$$x_1 = \min\left\{\frac{24}{6}, \frac{6}{1}, -, -\right\} = \min\{4, 6, -, -\} = 4$$

and $\mathbf{P}_3$ becomes the leaving vector.

The results above can be summarized in the familiar simplex tableau format. The presentation should help convince you that the two methods are essentially the same. You will find it instructive to develop similar tableaus in the succeeding iterations.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-5	-4	0	0	0	0	0
x_3	6						24
x_4	1						6
x_5	-1						1
x_6	0						2

Iteration 1

$$\mathbf{X}_{B_1} = (x_1, x_4, x_5, x_6), \mathbf{C}_{B_1} = (5, 0, 0, 0)$$
$$\mathbf{B}_1 = (\mathbf{P}_1, \mathbf{P}_4, \mathbf{P}_5, \mathbf{P}_6)$$
$$= \begin{pmatrix} 6 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

By using an appropriate inversion method (see Section D.2.7, in particular the *product form* method), the inverse is given as

$$\mathbf{B}_1^{-1} = \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{6} & 1 & 0 & 0 \\ \frac{1}{6} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$\mathbf{X}_{B_1} = \mathbf{B}_1^{-1}\mathbf{b} = (4, 2, 5, 2)^T,\ z = \mathbf{C}_{B_1}\mathbf{X}_{B_1} = 20$$

Optimality computations:

$$\mathbf{C}_{B_1}\mathbf{B}_1^{-1} = \left(\tfrac{5}{6}, 0, 0, 0\right)$$

$$\{z_j - c_j\}_{j=2,3} = \mathbf{C}_{B_1}\mathbf{B}_1^{-1}(\mathbf{P}_2, \mathbf{P}_3) - (c_2, c_3) = \left(-\tfrac{2}{3}, \tfrac{5}{6}\right)$$

Thus, $\mathbf{P}_2$ is the entering vector.

Feasibility computations:

$$\mathbf{X}_{B_1} = (x_1, x_4, x_5, x_6)^T = (4, 2, 5, 2)^T$$

$$\mathbf{B}_1^{-1}\mathbf{P}_2 = \left(\tfrac{2}{3}, \tfrac{4}{3}, \tfrac{5}{3}, 1\right)^T$$

Hence,

$$x_2 = \min\left\{\frac{4}{\frac{2}{3}}, \frac{2}{\frac{4}{3}}, \frac{5}{\frac{5}{3}}, \frac{2}{1}\right\} = \min\left\{6, \tfrac{3}{2}, 3, 2\right\} = \tfrac{3}{2}$$

and $\mathbf{P}_4$ becomes the leaving vector. (You will find it helpful to summarize the results above in the simplex tableau format as we did in iteration 0.)

Iteration 2

$$\mathbf{X}_{B_2} = (x_1, x_2, x_5, x_6)^T,\ \mathbf{C}_{B_2} = (5, 4, 0, 0)$$

$$\mathbf{B}_2 = (\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_5, \mathbf{P}_6)$$

$$= \begin{pmatrix} 6 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Hence,

$$\mathbf{B}_2^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & \frac{3}{4} & 0 & 0 \\ \frac{3}{8} & -\frac{5}{4} & 1 & 0 \\ \frac{1}{8} & -\frac{3}{4} & 0 & 1 \end{pmatrix}$$

Thus,

$$\mathbf{X}_{B_2} = \mathbf{B}_2^{-1}\mathbf{b} = \left(3, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{1}{2}\right)^T, z = \mathbf{C}_{B_2}\mathbf{X}_{B_2} = 21$$

Optimality computations:

$$\mathbf{C}_{B_2}\mathbf{B}_2^{-1} = \left(\tfrac{3}{4}, \tfrac{1}{2}, 0, 0\right)$$

$$\{z_j - c_j\}_{j=3,4} = \mathbf{C}_{B_2}\mathbf{B}_2^{-1}(\mathbf{P}_3, \mathbf{P}_4) - (c_3, c_4) = \left(\tfrac{3}{4}, \tfrac{1}{2}\right)$$

Thus, $\mathbf{X}_{B_2}$ is optimal and the computations end.

Summary of optimal solution:

$$x_1 = 3, x_2 = 1.5, z = 21$$

PROBLEM SET 7.2B

1. In Example 7.2-1, summarize the data of iteration 1 in the tableau format of Section 3.3.
2. Solve the following LPs by the revised simplex method:
   (a) Maximize $z = 6x_1 - 2x_2 + 3x_3$
   subject to

$$\begin{aligned} 2x_1 - x_2 + 2x_3 &\le 2 \\ x_1 \qquad + 4x_3 &\le 4 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

   ***(b)** Maximize $z = 2x_1 + x_2 + 2x_3$
   subject to

$$\begin{aligned} 4x_1 + 3x_2 + 8x_3 &\le 12 \\ 4x_1 + x_2 + 12x_3 &\le 8 \\ 4x_1 - x_2 + 3x_3 &\le 8 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

   (c) Minimize $z = 2x_1 + x_2$
   subject to

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\ge 6 \\ x_1 + 2x_2 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

(d) Minimize $z = 5x_1 - 4x_2 + 6x_3 + 8x_4$

subject to

$$\begin{aligned} x_1 + 7x_2 + 3x_3 + 7x_4 &\le 46 \\ 3x_1 - x_2 + x_3 + 2x_4 &\le 20 \\ 2x_1 + 3x_2 - x_3 + x_4 &\ge 18 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

3. Solve the following LP by the revised simplex method given the starting basic feasible vector $\mathbf{X}_{B_0} = (x_2, x_4, x_5)^T$.

$$\text{Minimize } z = 7x_2 + 11x_3 - 10x_4 + 26x_6$$

subject to

$$\begin{aligned} x_2 - x_3 \quad\quad + x_5 + x_6 &= 6 \\ x_2 - x_3 + x_4 \quad\quad + 3x_6 &= 8 \\ x_1 + x_2 - 3x_3 + x_4 + x_5 \quad\quad &= 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\ge 0 \end{aligned}$$

4. Solve the following using the two-phase revised simplex method:

(a) Problem 2-c.

(b) Problem 2-d.

(c) Problem 3 (ignore the given starting $\mathbf{X}_{B_0}$).

5. *Revised Dual Simplex Method.* The steps of the revised dual simplex method (using matrix manipulations) can be summarized as follows:

Step 0. Let $\mathbf{B}_0 = \mathbf{I}$ be the starting basis for which at least one of the elements of $\mathbf{X}_{B_0}$ is negative (infeasible).

Step 1. Compute $\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b}$, the current values of the basic variables. Select the leaving variable x_r as the one having the most negative value. If all the elements of $\mathbf{X}_B$ are nonnegative, stop; the current solution is feasible (and optimal).

Step 2. **(a)** Compute $z_j - c_j = \mathbf{C}_B\mathbf{B}^{-1}\mathbf{P}_j - c_j$ for all the nonbasic variables x_j.

(b) For all the nonbasic variables x_j, compute the constraint coefficients $(\mathbf{B}^{-1}\mathbf{P}_j)_r$ associated with the row of the leaving variable x_r.

(c) The entering variable is associated with

$$\theta = \min_i \left\{ \left| \frac{z_j - c_j}{(\mathbf{B}^{-1}\mathbf{P}_j)_r} \right|, (\mathbf{B}^{-1}\mathbf{P}_j)_r < 0 \right\}$$

If all $(\mathbf{B}^{-1}\mathbf{P}_j)_r \ge 0$, no feasible solution exists.

Step 3. Obtain the new basis by interchanging the entering and leaving vectors ($\mathbf{P}_j$ and $\mathbf{P}_r$). Compute the new inverse and go to step 1.

Apply the method to the following problem:

$$\text{Minimize } z = 3x_1 + 2x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 &\ge 3 \\ 4x_1 + 3x_2 &\ge 6 \\ x_1 + 2x_2 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

7.3 BOUNDED-VARIABLES ALGORITHM

In LP models, variables may have explicit positive upper and lower bounds. For example, in production facilities, lower and upper bounds can represent the minimum and maximum demands for certain products. Bounded variables also arise prominently in the course of solving integer programming problems by the branch-and-bound algorithm (see Section 9.3.1).

The bounded algorithm is efficient computationally because it accounts for the bounds *implicitly*. We consider the lower bounds first because it is simpler. Given $\mathbf{X} \geq \mathbf{L}$, we can use the substitution

$$\mathbf{X} = \mathbf{L} + \mathbf{X}', \qquad \mathbf{X}' \geq \mathbf{0}$$

throughout and solve the problem in terms of $\mathbf{X}'$ (whose lower bound now equals zero). The original $\mathbf{X}$ is determined by back-substitution, which is legitimate because it guarantees that $\mathbf{X} = \mathbf{X}' + \mathbf{L}$ will remain nonnegative for all $\mathbf{X}' \geq \mathbf{0}$.

Next, consider the upper bounding constraints, $\mathbf{X} \leq \mathbf{U}$. The idea of direct substitution (i.e., $\mathbf{X} = \mathbf{U} - \mathbf{X}''$, $\mathbf{X}'' \geq \mathbf{0}$) is not correct because back-substitution, $\mathbf{X} = \mathbf{U} - \mathbf{X}''$, does not ensure that $\mathbf{X}$ will remain nonnegative. A different procedure is thus needed.

Define the upper bounded LP model as

$$\text{Maximize } z = \{\mathbf{CX} \mid (\mathbf{A}, \mathbf{I})\mathbf{X} = \mathbf{b}, \mathbf{0} \leq \mathbf{X} \leq \mathbf{U}\}$$

The bounded algorithm uses only the constraints $(\mathbf{A}, \mathbf{I})\mathbf{X} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$, while accounting for $\mathbf{X} \leq \mathbf{U}$ implicitly by modifying the simplex feasibility condition.

Let $\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b}$ be a current basic feasible solution of $(\mathbf{A}, \mathbf{I})\mathbf{X} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$, and suppose that, according to the (regular) optimality condition, $\mathbf{P}_j$ is the entering vector. Then, *given that all the nonbasic variables are zero,* the constraint equation of the ith basic variable can be written as

$$(\mathbf{X}_B)_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{P}_j)_i\, x_j$$

When the entering variable x_j increases above zero level, $(\mathbf{X}_B)_i$ will *increase or decrease* depending on whether $(\mathbf{B}^{-1}\mathbf{P}_j)_i$ is negative or positive, respectively. Thus, in determining the value of the entering variable x_j, three conditions must be satisfied.

1. The basic variable $(\mathbf{X}_B)_i$ remains nonnegative—that is, $(\mathbf{X}_B)_i \geq 0$.
2. The basic variable $(\mathbf{X}_B)_i$ does not exceed its upper bound—that is, $(\mathbf{X}_B)_i \leq (\mathbf{U}_B)_i$, where $\mathbf{U}_B$ comprises the ordered elements of $\mathbf{U}$ corresponding to $\mathbf{X}_B$.
3. The entering variable x_j cannot assume a value larger than its upper bound—that is, $x_j \leq u_j$, where u_j is the jth element of $\mathbf{U}$.

The first condition $(\mathbf{X}_B)_i \geq 0$ requires that

$$(\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{P}_j)_i\, x_j \geq 0$$

It is satisfied if

$$x_j \leq \theta_1 = \min_i \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{P}_j)_i} \,\middle|\, (\mathbf{B}^{-1}\mathbf{P}_j)_i > 0 \right\}$$

This condition is the same as the feasibility condition of the regular simplex method.

Next, the condition $(\mathbf{X}_B)_i \leq (\mathbf{U}_B)_i$ specifies that

$$(\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{P}_j)_i\, x_j \leq (\mathbf{U}_B)_i$$

It is satisfied if

$$x_j \leq \theta_2 = \min_i \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{U}_B)_i}{(\mathbf{B}^{-1}\mathbf{P}_j)_i} \,\middle|\, (\mathbf{B}^{-1}\mathbf{P}_j)_i < 0 \right\}$$

Combining the three restrictions, x_j enters the solution at the level that satisfies all three conditions—that is,

$$x_j = \min\{\theta_1, \theta_2, u_j\}$$

The change of basis for the next iteration depends on whether x_j enters the solution at level θ_1, θ_2, or u_j. Assuming that $(\mathbf{X}_B)_r$ is the leaving variable, then we have the following rules:

1. $x_j = \theta_1$: $(\mathbf{X}_B)_r$ leaves the basic solution (becomes nonbasic) at level zero. The new iteration is generated using the normal simplex method with x_j and $(\mathbf{X}_B)_r$ as the entering and the leaving variables, respectively.
2. $x_j = \theta_2$: $(\mathbf{X}_B)_r$ becomes nonbasic *at its upper bound*. The new iteration is generated as in the case of $x_j = \theta_1$, with one modification that accounts for the fact that $(\mathbf{X}_B)_r$ will be nonbasic at *upper bound*. Because the values of θ_1 and θ_2 require *all nonbasic variables to be at zero level* (convince yourself that this is the case!), we must convert the new nonbasic $(\mathbf{X}_B)_r$ at upper bound to a nonbasic variable at zero level. This is achieved by using the substitution $(\mathbf{X}_B)_r = (\mathbf{U}_B)_r - (\mathbf{X}'_B)_r$, where $(\mathbf{X}'_B)_r \geq 0$. It is immaterial whether the substitution is made before or after the new basis is computed.
3. $x_j = u_j$: The basic vector $\mathbf{X}_B$ remains unchanged because $x_j = u_j$ stops short of forcing any of the current basic variables to reach its lower $(= 0)$ or upper bound. This means that x_j will remain nonbasic *but at upper bound*. Following the argument just presented, the new iteration is generated by using the substitution $x_j = u_j - x'_j$.

A tie among θ_1, θ_2, and u_j may be broken arbitrarily. However, it is preferable, where possible, to implement the rule for $x_j = u_j$ because it entails less computation.

The substitution $x_j = u_j - x'_j$ will change the original c_j, $\mathbf{P}_j$, and $\mathbf{b}$ to $c'_j = -c_j$, $\mathbf{P}'_j = -\mathbf{P}_j$, and $\mathbf{b}$ to $\mathbf{b}' = \mathbf{b} - u_j\mathbf{P}_j$. This means that if the revised simplex method is used, all the computations (e.g., $\mathbf{B}^{-1}$, $\mathbf{X}_B$, and $z_j - c_j$), should be based on the updated values of $\mathbf{C}$, $\mathbf{A}$, and $\mathbf{b}$ at each iteration (see Problem 5, Set 7.3a, for further details).

Example 7.3-1

Solve the following LP model by the upper-bounding algorithm.[2]

$$\text{Maximize } z = 3x_1 + 5y + 2x_3$$

[2]You can use TORA's Linear Programming ⇒ Solve problem ⇒ Algebraic ⇒ Iterations ⇒ Bounded simplex to produce the associated simplex iterations (file toraEx7.3-1.txt).

subject to

$$x_1 + \ \ y + 2x_3 \leq 14$$
$$2x_1 + 4y + 3x_3 \leq 43$$
$$0 \leq x_1 \leq 4, 7 \leq y \leq 10, 0 \leq x_3 \leq 3$$

The lower bound on y is accounted for using the substitution $y = x_2 + 7$, where $0 \leq x_2 \leq 10 - 7 = 3$.

To avoid being "sidetracked" by the computational details, we will not use the revised simplex method to carry out the computations. Instead, we will use the compact tableau form. Problems 5, 6, and 7, Set 7.3a address the revised version of the algorithm.

Iteration 0

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	-3	-5	-2	0	0	35
x_4	1	1	2	1	0	7
x_5	2	4	3	0	1	15

We have $\mathbf{B} = \mathbf{B}^{-1} = \mathbf{I}$ and $\mathbf{X}_B = (x_4, x_5)^T = \mathbf{B}^{-1}\mathbf{b} = (7, 15)^T$. Given that x_2 is the entering variable ($z_2 - c_2 = -5$), we get

$$\mathbf{B}^{-1}\mathbf{P}_2 = (1, 4)^T$$

which yields

$$\theta_1 = \min\left\{\frac{7}{1}, \frac{15}{4}\right\} = 3.75, \text{corresponding to } x_5$$

$$\theta_2 = \infty (\text{because all the elements of } \mathbf{B}^{-1}\mathbf{P}_2 > \mathbf{0})$$

Next, given the upper bound on the entering variable, $x_2 \leq 3$, it follows that

$$x_2 = \min\{3.75, \infty, 3\}$$
$$= 3 \ (= u_2)$$

Because x_2 enters at its upper bound ($= u_2 = 3$), $\mathbf{X}_B$ remains unchanged, and x_2 becomes nonbasic *at its upper bound.* We use the substitution $x_2 = 3 - x_2'$ to obtain the new tableau as

Basic	x_1	x_2'	x_3	x_4	x_5	Solution
z	-3	5	-2	0	0	50
x_4	1	-1	2	1	0	4
x_5	2	-4	3	0	1	3

The substitution in effect changes the original right-hand side vector from $\mathbf{b} = (7, 15)^T$ to $\mathbf{b}' = (4, 3)^T$. This change should be considered in future computations.

Iteration 1. The entering variable is x_1. The basic vector $\mathbf{X}_B$ and $\mathbf{B}^{-1}$ $(= \mathbf{I})$ are the same as in iteration 0. Next,

$$\mathbf{B}^{-1}\mathbf{P}_1 = (1, 2)^T$$

$$\theta_1 = \min\left\{\frac{4}{1}, \frac{3}{2}\right\} = 1.5, \text{ corresponding to basic } x_5$$

$$\theta_2 = \infty(\text{because } \mathbf{B}^{-1}\mathbf{P}_1 > \mathbf{0})$$

Thus,

$$x_1 = \min\{1.5, \infty, 4\}$$
$$= 1.5 \; (= \theta_1)$$

Thus, the entering variable x_1 becomes basic, and the leaving variable x_5 becomes nonbasic at zero level, which yields

Basic	x_1	x_2'	x_3	x_4	x_5	Solution
z	0	-1	$\frac{5}{2}$	0	$\frac{3}{2}$	$\frac{109}{2}$
x_4	0	1	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{5}{2}$
x_1	1	-2	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{3}{2}$

Iteration 2 The new inverse is

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Now

$$\mathbf{X}_B = (x_4, x_1)^T = \mathbf{B}^{-1}\mathbf{b}' = \left(\tfrac{5}{2}, \tfrac{3}{2}\right)^T$$

where $\mathbf{b}' = (4, 3)^T$ as computed at the end of iteration 0. We select x_2' as the entering variable, and, noting that $\mathbf{P}_2' = -\mathbf{P}_2$, we get

$$\mathbf{B}^{-1}\mathbf{P}_2' = (1, -2)^T$$

Thus,

$$\theta_1 = \min\left\{\frac{\frac{5}{2}}{1}, -\right\} = 2.5, \text{ corresponding to basic } x_4$$

$$\theta_2 = \min\left\{-, \frac{\frac{3}{2} - 4}{-2}\right\} = 1.25, \text{ corresponding to basic } x_1$$

We then have

$$x_2' = \min\{2.5, 1.25, 3\}$$
$$= 1.25 \; (= \theta_2)$$

Because x_1 becomes nonbasic at its upper bound, we apply the substitution $x_1 = 4 - x_1'$ to obtain

Basic	x_1'	x_2'	x_3	x_4	x_5	Solution
z	0	-1	$\frac{5}{2}$	0	$\frac{3}{2}$	$\frac{109}{2}$
x_4	0	1	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{5}{2}$
x_1'	-1	-2	$\frac{3}{2}$	0	$\frac{1}{2}$	$-\frac{5}{2}$

Next, the entering variable x_2' becomes basic and the leaving variable x_1 becomes nonbasic at upper bound, which yields

Basic	x_1'	x_2'	x_3	x_4	x_5	Solution
z	$\frac{1}{2}$	0	$\frac{7}{4}$	0	$\frac{5}{4}$	$\frac{223}{4}$
x_4	$-\frac{1}{2}$	0	$\frac{5}{4}$	1	$-\frac{1}{4}$	$\frac{5}{4}$
x_2'	$\frac{1}{2}$	1	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{5}{4}$

The last tableau is feasible and optimal. Note that the last two steps could have been reversed—meaning that we could first make x_2' basic and then apply the substitution $x_1 = 4 - x_1'$ (try it!). The sequence presented here involves less computation, however.

The optimal values of x_1, x_2, and x_3 are obtained by back-substitution as $x_1 = u_1 - x_1' = 4 - 0 = 4$, $x_2 = u_2 - x_2' = 3 - \frac{5}{4} = \frac{7}{4}$, and $x_3 = 0$. Finally, we get $y = l_2 + x_2 = 7 + \frac{7}{4} = \frac{35}{4}$. The associated optimal value of the objective function z is $\frac{223}{4}$.

PROBLEM SET 7.3A

1. Consider the following linear program:

   Maximize $z = 2x_1 + x_2$

   subject to

$$x_1 + x_2 \le 3$$
$$0 \le x_1 \le 2, 0 \le x_2 \le 2$$

   (a) Solve the problem graphically, and trace the sequence of extreme points leading to the optimal solution. (You may use TORA.)

   (b) Solve the problem by the upper-bounding algorithm and show that the method produces the same sequence of extreme points as in the graphical optimal solution (you may use TORA to generate the iterations).

   (c) How does the upper-bounding algorithm recognize the extreme points?

*2. Solve the following problem by the bounded algorithm:

$$\text{Maximize } z = 6x_1 + 2x_2 + 8x_3 + 4x_4 + 2x_5 + 10x_6$$

subject to

$$8x_1 + x_2 + 8x_3 + 2x_4 + 2x_5 + 4x_6 \le 13$$
$$0 \le x_j \le 1, j = 1, 2, \ldots, 6$$

3. Solve the following problems by the bounded algorithm:

   (a) Minimize $z = 6x_1 - 2x_2 - 3x_3$

   subject to

$$2x_1 + 4x_2 + 2x_3 \leq 8$$
$$x_1 - 2x_2 + 3x_3 \leq 7$$
$$0 \leq x_1 \leq 2, 0 \leq x_2 \leq 2, 0 \leq x_3 \leq 1$$

   (b) Maximize $z = 3x_1 + 5x_2 + 2x_3$

   subject to

$$x_1 + 2x_2 + 2x_3 \leq 10$$
$$2x_1 + 4x_2 + 3x_3 \leq 15$$
$$0 \leq x_1 \leq 4, 0 \leq x_2 \leq 3, 0 \leq x_3 \leq 3$$

4. In the following problems, some of the variables have positive lower bounds. Use the bounded algorithm to solve these problems.

   (a) Maximize $z = 3x_1 + 2x_2 - 2x_3$

   subject to

$$2x_1 + x_2 + x_3 \leq 8$$
$$x_1 + 2x_2 - x_3 \geq 3$$
$$1 \leq x_1 \leq 3, 0 \leq x_2 \leq 3, 2 \leq x_3$$

   (b) Maximize $z = x_1 + 2x_2$

   subject to

$$-x_1 + 2x_2 \geq 0$$
$$3x_1 + 2x_2 \leq 10$$
$$-x_1 + x_2 \leq 1$$
$$1 \leq x_1 \leq 3, 0 \leq x_2 \leq 1$$

   (c) Maximize $z = 4x_1 + 2x_2 + 6x_3$

   subject to

$$4x_1 - x_2 \leq 9$$
$$-x_1 + x_2 + 2x_3 \leq 8$$
$$-3x_1 + x_2 + 4x_3 \leq 12$$
$$1 \leq x_1 \leq 3, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 2$$

5. Consider the matrix definition of the bounded-variables problem. Suppose that the vector $\mathbf{X}$ is partitioned into $(\mathbf{X}_z, \mathbf{X}_u)$, where $\mathbf{X}_u$ represents the basic *and* nonbasic variables that will be substituted at upper bound during the course of the algorithm. The problem may thus be written as

$$\begin{pmatrix} 1 & -\mathbf{C}_z & -\mathbf{C}_u \\ 0 & \mathbf{D}_z & \mathbf{D}_u \end{pmatrix} \begin{pmatrix} z \\ \mathbf{X}_z \\ \mathbf{X}_u \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}$$

Using $\mathbf{X}_u = \mathbf{U}_u - \mathbf{X}'_u$ where $\mathbf{U}_u$ is a subset of $\mathbf{U}$ representing the upper bounds for $\mathbf{X}_u$, let $\mathbf{B}$ (and $\mathbf{X}_B$) be the basis of the current simplex iteration after $\mathbf{X}_u$ has been substituted out. Show that the associated general simplex tableau is given as

Basic	$\mathbf{X}_z^T$	$\mathbf{X}'^T_u$	Solution
z	$\mathbf{C}_B\mathbf{B}^{-1}\mathbf{D}_z - \mathbf{C}_z$	$-\mathbf{C}_B\mathbf{B}^{-1}\mathbf{D}_u + \mathbf{C}_u$	$\mathbf{C}_u\mathbf{B}^{-1}\mathbf{b}' + \mathbf{C}_u\mathbf{U}_u$
$\mathbf{X}_B$	$\mathbf{B}^{-1}\mathbf{D}_z$	$-\mathbf{B}^{-1}\mathbf{D}_u$	$\mathbf{B}^{-1}\mathbf{b}'$

where $\mathbf{b}' = \mathbf{b} - \mathbf{D}_u\mathbf{U}_u$.

6. In Example 7.3-1, do the following:
   (a) In Iteration 1, verify that $\mathbf{X}_B = (x_4, x_1)^T = \left(\frac{5}{2}, \frac{3}{2}\right)^T$ by using matrix manipulation.
   (b) In Iteration 2, show how $\mathbf{B}^{-1}$ can be computed from the original data of the problem. Then verify the given values of basic x_4 and x'_2 using matrix manipulation.
7. Solve part (a) of Problem 3 using the revised simplex (matrix) version for upper-bounded variables.
8. *Bounded Dual Simplex Algorithm.* The dual simplex algorithm (Section 4.4.1) can be modified to accommodate the bounded variables as follows. Given the upper bound constraint $x_j \le u_j$ for all j (if u_j is infinite, replace it with a sufficiently large upper bound M), the LP problem is converted to a dual feasible (i.e., primal optimal) form by using the substitution $x_j = u_j - x'_j$, where necessary.

   Step 1. If any of the current basic variables $(\mathbf{X}_B)_i$ exceeds its upper bound, use the substitution $(\mathbf{X}_B)_i = (\mathbf{U}_B)_i - (\mathbf{X}_B)'_i$. Go to step 2.

   Step 2. If all the basic variables are feasible, stop. Otherwise, select the leaving variable x_r as the basic variable having the most negative value. Go to step 3.

   Step 3. Select the entering variable using the optimality condition of the regular dual simplex method (Section 4.4.1). Go to step 4.

   Step 4. Perform a change of basis. Go to step 1.

   Apply the given algorithm to the following problems:

   (a) Minimize $z = -3x_1 - 2x_2 + 2x_3$
   subject to
$$2x_1 + x_2 + x_3 \le 8$$
$$-x_1 + 2x_2 + x_3 \ge 13$$
$$0 \le x_1 \le 2, 0 \le x_2 \le 3, 0 \le x_3 \le 1$$

   (b) Maximize $z = x_1 + 5x_2 - 2x_3$
   subject to
$$4x_1 + 2x_2 + 2x_3 \le 26$$
$$x_1 + 3x_2 + 4x_3 \ge 17$$
$$0 \le x_1 \le 2, 0 \le x_2 \le 3, x_3 \ge 0$$

7.4 DUALITY

We have dealt with the dual problem in Chapter 4. This section presents a more rigorous treatment of duality and allows us to verify the primal-dual relationships that formed the basis for post-optimal analysis in Chapter 4. The presentation also lays the foundation for the development of parametric programming.

7.4.1 Matrix Definition of the Dual Problem

Suppose that the primal problem in equation form with m constraints and n variables is defined as

$$\text{Maximize } z = \mathbf{CX}$$

subject to

$$\mathbf{AX} = \mathbf{b}$$
$$\mathbf{X} \geq \mathbf{0}$$

Letting the vector $\mathbf{Y} = (y_1, y_2, \ldots, y_m)$ represent the dual variables, the rules in Table 4.2 produce the following dual problem:

$$\text{Minimize } w = \mathbf{Yb}$$

subject to

$$\mathbf{YA} \geq \mathbf{C}$$
$$\mathbf{Y} \text{ unrestricted}$$

Some of the constraints $\mathbf{YA} \geq \mathbf{C}$ may override unrestricted $\mathbf{Y}$.

PROBLEM SET 7.4A

1. Prove that the dual of the dual is the primal.

*2. If the primal is given as min $z = \{\mathbf{CX} | \mathbf{AX} \geq \mathbf{b}, \mathbf{X} \geq \mathbf{0}\}$, define the corresponding dual problem.

7.4.2 Optimal Dual Solution

This section establishes relationships between the primal and dual problems and shows how the optimal dual solution can be determined from the optimal primal solution. Let $\mathbf{B}$ be the current *optimal* primal basis, and define $\mathbf{C}_B$ as the objective-function coefficients associated with the optimal vector $\mathbf{X}_B$.

Theorem 7.4-1. (Weak Duality Theory). *For any pair of* feasible *primal and dual solutions* $(\mathbf{X}, \mathbf{Y})$, *the value of the objective function in the* minimization *problem sets an upper bound on the value of the objective function in the* maximization *problem. For the* optimal *pair* $(\mathbf{X}^*, \mathbf{Y}^*)$, *the values of the objective functions are equal.*

Proof. The feasible pair $(\mathbf{X}, \mathbf{Y})$ satisfies all the restrictions of the two problems. Premultiplying both sides of the constraints of the maximization problem with (unrestricted) $\mathbf{Y}$, we get

$$\mathbf{YAX} = \mathbf{YB} = w \tag{1}$$

Also, for the minimization problem, postmultiplying both sides of each of the first two sets of constraints by $\mathbf{X}(\geq \mathbf{0})$, we get

$$\mathbf{YAX} \geq \mathbf{CX}$$

or

$$\mathbf{YAX} \geq \mathbf{CX} = z \tag{2}$$

(The nonnegativity of the vector **X** is essential for preserving the direction of the inequality.) Combining (1) and (2), we get $z \leq w$ for any *feasible* pair $(\mathbf{X}, \mathbf{Y})$.

Note that the theorem does *not* depend on labeling the problems as primal or dual. What is important is the sense of optimization in each problem. Specifically, for any pair of feasible solutions, the objective value in the maximization problem does not exceed the objective value in the minimization problem.

The implication of the theorem is that, given $z \leq w$ for any feasible solutions, the maximum of z and the minimum of w are achieved when the two objective values are equal. A consequence of this result is that the "goodness" of any feasible primal and dual solutions relative to the optimum may be checked by comparing the difference $(w - z)$ to $\frac{z + w}{2}$. The smaller the ratio $\frac{2(w - z)}{z + w}$, the closer the two solutions are to being optimal. The suggested *rule of thumb* does *not* imply that the optimal objective value is $\frac{z + w}{2}$.

What happens if one of the two problems has an unbounded objective value? The answer is that the other problem must be infeasible. For if it is not, then both problems have feasible solutions, and the relationship $z \leq w$ must hold—an impossible result, because either $z = +\infty$ or $w = -\infty$ by assumption.

The next question is: If one problem is infeasible, is the other problem unbounded? Not necessarily. The following counterexample shows that both the primal and the dual can be infeasible (verify graphically!):

Primal. Maximize $z = \{x_1 + x_2 | x_1 - x_2 \leq -1, -x_1 + x_2 \leq -1, x_1, x_2 \geq 0\}$

Dual. Minimize $w = \{-y_1 - y_2 | y_1 - y_2 \geq 1, -y_1 + y_2 \geq 1, y_1, y_2 \geq 0\}$

Theorem 7.4-2. *Given the* optimal *primal basis* **B** *and its associated objective coefficient vector* $\mathbf{C}_B$, *the optimal solution of the dual problem is*

$$\mathbf{Y} = \mathbf{C}_B\mathbf{B}^{-1}$$

Proof. The proof rests on verifying two points: $\mathbf{Y} = \mathbf{C}_B\mathbf{B}^{-1}$ is a feasible dual solution and $z = w$, per Theorem 7.4-1.

The feasibility of $\mathbf{Y} = \mathbf{C}_B\mathbf{B}^{-1}$ is guaranteed by the optimality of the primal, $z_j - c_j \geq 0$ for all j—that is,

$$\mathbf{C}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{C} \geq \mathbf{0}$$

(See Section 7.2.1.) Thus, $\mathbf{YA} - \mathbf{C} \geq \mathbf{0}$ or $\mathbf{YA} \geq \mathbf{C}$, which shows that $\mathbf{Y} = \mathbf{C}_B\mathbf{B}^{-1}$ is a feasible dual solution.

Next, we show that the associated $w = z$ by noting that

$$w = \mathbf{Yb} = \mathbf{C}_B\mathbf{B}^{-1}\mathbf{b} \tag{1}$$

Similarly, given the primal solution $\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b}$, we get

$$z = \mathbf{C}_B\mathbf{X}_B = \mathbf{C}_B\mathbf{B}^{-1}\mathbf{b} \tag{2}$$

From relations (1) and (2), we conclude that $z = w$.

The dual variables $\mathbf{Y} = \mathbf{C}_B\mathbf{B}^{-1}$ are sometimes referred to as the **dual** or **shadow prices**, names that evolved from the economic interpretation of the dual variables in Section 4.3.1.

Given that $\mathbf{P}_j$ is the jth column of $\mathbf{A}$, we note from Theorem 7.4-2 that

$$z_j - c_j = \mathbf{C}_B\mathbf{B}^{-1}\mathbf{P}_j - c_j = \mathbf{Y}\mathbf{P}_j - c_j$$

represents the difference between the left- and right-hand sides of the dual constraints. The maximization primal starts with $z_j - c_j < 0$ for at least one j, which means that the corresponding dual constraint, $\mathbf{Y}\mathbf{P}_j \geq c_j$, is not satisfied. When the primal optimal is reached we get $z_j - c_j \geq 0$, for all j, which means that the corresponding dual solution $\mathbf{Y} = \mathbf{C}_B\mathbf{B}^{-1}$ becomes feasible. Thus, while the primal is seeking optimality, the dual is automatically seeking feasibility. This point is the basis for the development of the *dual simplex method* (Section 4.4.1) in which the iterations start better than optimal and infeasible and remain so until feasibility is acquired at the last iteration. This is in contrast with the (primal) simplex method (Chapter 3), which remains worse than optimal but feasible until the optimal iteration is reached.

Example 7.4-1

The *optimal* basis for the following LP is $\mathbf{B} = (\mathbf{P}_1, \mathbf{P}_4)$. Write the dual and find its optimum solution using the optimal primal basis.

$$\text{Maximize } z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 \quad\;\; &= 5 \\ -x_1 + 3x_2 \quad\;\; + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The dual problem is

$$\text{Minimize } w = 5y_1 + 2y_2$$

subject to

$$\begin{aligned} y_1 - y_2 &\geq 3 \\ 2y_1 + 3y_2 &\geq 5 \\ y_1, y_2 &\geq 0 \end{aligned}$$

We have $\mathbf{X}_B = (x_1, x_4)^T$ and $\mathbf{C}_B = (3, 0)$. The optimal basis and its inverse are

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

The associated primal and dual values are

$$(x_1, x_4)^T = \mathbf{B}^{-1}\mathbf{b} = (5, 7)^T$$

$$(y_1, y_2) = \mathbf{C}_B\mathbf{B}^{-1} = (3, 0)$$

Both solutions are feasible and $z = w = 15$ (verify!). Thus, the two solutions are optimal.

PROBLEM SET 7.4B

1. Verify that the dual problem of the numeric example given at the end of Theorem 7.4-1 is correct. Then verify graphically that both the primal and dual problems have no feasible solution.
2. Consider the following LP:

$$\text{Maximize } z = 50x_1 + 30x_2 + 10x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 \quad &= 1 \\ 2x_2 \quad &= -5 \\ 4x_1 \quad + x_3 &= 6 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(a) Write the dual.
(b) Show by inspection that the primal is infeasible.
(c) Show that the dual in (a) is unbounded.
(d) From Problems 1 and 2, develop a general conclusion regarding the relationship between infeasibility and unboundedness in the primal and dual problems.

3. Consider the following LP:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$\begin{aligned} 2x_1 - x_2 + 3x_3 \quad &= 2 \\ x_1 + 2x_2 + x_3 + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(a) Write the dual.
(b) In each of the following cases, first verify that the given basis $\mathbf{B}$ is feasible for the primal. Next, using $\mathbf{Y} = \mathbf{C}_B\mathbf{B}^{-1}$, compute the associated dual values and verify whether or not the primal solution is optimal.

(i) $\mathbf{B} = (\mathbf{P}_4, \mathbf{P}_3)$ **(iii)** $\mathbf{B} = (\mathbf{P}_1, \mathbf{P}_2)$
(ii) $\mathbf{B} = (\mathbf{P}_2, \mathbf{P}_3)$ **(iv)** $\mathbf{B} = (\mathbf{P}_1, \mathbf{P}_4)$

4. Consider the following LP:

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 4x_2 + \quad + x_4 &= 8 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(a) Write the dual problem.
(b) Verify that $\mathbf{B} = (\mathbf{P}_2, \mathbf{P}_3)$ is optimal by computing $z_j - c_j$ for all nonbasic $\mathbf{P}_j$.
(c) Find the associated optimal dual solution.

***5.** An LP model includes two variables x_1 and x_2 and three constraints of the type $\leq$. The associated slacks are x_3, x_4, and x_5. Suppose that the optimal basis is $\mathbf{B} = (\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$, and its inverse is

$$\mathbf{B}^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

The optimal primal and dual solutions are

$$\mathbf{X}_B = (x_1, x_2, x_3)^T = (2, 6, 2)^T$$
$$\mathbf{Y} = (y_1, y_2, y_3) = (0, 3, 2)$$

Determine the optimal value of the objective function in two ways using the primal and dual problems.

6. Prove the following relationship for the optimal primal and dual solutions:

$$\sum_{i=1}^{m} c_i(\mathbf{B}^{-1}\mathbf{P}_k)_i = \sum_{i=1}^{m} y_i a_{ik}$$

where $\mathbf{C}_B = (c_1, c_2, \ldots, c_m)$ and $\mathbf{P}_k = (a_{1k}, a_{2k}, \ldots, a_{mk})^T$, for $k = 1, 2, \ldots, n$, and $(\mathbf{B}^{-1}\mathbf{P}_k)_i$ is the ith element of $\mathbf{B}^{-1}\mathbf{P}_k$.

***7.** Write the dual of

$$\text{Maximize } z = \{\mathbf{CX} | \mathbf{AX} = \mathbf{b}, \quad \mathbf{X} \text{ unrestricted}\}$$

8. Show that the dual of

$$\text{Maximize } z = \{\mathbf{CX} | \mathbf{AX} \leq \mathbf{b}, \quad \mathbf{0} < \mathbf{L} \leq \mathbf{X} \leq \mathbf{U}\}$$

always possesses a feasible solution.

7.5 PARAMETRIC LINEAR PROGRAMMING

Parametric linear programming is an extension of the post-optimal analysis presented in Section 4.5. It investigates the effect of *predetermined* continuous variations in the objective function coefficients and the right-hand side of the constraints on the optimum solution.

Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ and define the LP as

$$\text{Maximize } z = \left\{\mathbf{CX} | \sum_{j=1}^{n} \mathbf{P}_j x_j = \mathbf{b}, \mathbf{X} \geq \mathbf{0}\right\}$$

In parametric analysis, the objective function and right-hand side vectors, **C** and **b**, are replaced with the parameterized functions **C**(t) and **b**(t), where t is the parameter of variation. Mathematically, t can assume any positive or negative value. In practice, however, t usually represents time, and hence it is nonnegative. In this presentation we will assume $t \geq 0$.

The general idea of parametric analysis is to start with the optimal solution at $t = 0$. Then, using the optimality and feasibility conditions of the simplex method, we determine the range $0 \leq t \leq t_1$ for which the solution at $t = 0$ remains optimal and feasible. In this case, t_1 is referred to as a **critical value**. The process continues by determining successive critical values and their corresponding optimal feasible solutions, and will terminate at $t = t_r$ when there is indication that either the last solution remains unchanged for $t > t_r$ or that no feasible solution exists beyond that critical value.

7.5.1 Parametric Changes in C

Let $\mathbf{X}_{B_i}$, $\mathbf{B}_i$, $\mathbf{C}_{B_i}(t)$ be the elements that define the optimal solution associated with critical t_i (the computations start at $t_0 = 0$ with $\mathbf{B}_0$ as its optimal basis). Next, the critical value t_{i+1} and its optimal basis, if one exists, is determined. Because changes in **C** can affect only the optimality of the problem, the current solution $\mathbf{X}_{B_i} = \mathbf{B}_i^{-1}\mathbf{b}$ will remain optimal for some $t \geq t_i$ so long as the reduced cost, $z_j(t) - c_j(t)$, satisfies the following optimality condition:

$$z_j(t) - c_j(t) = \mathbf{C}_{B_i}(t)\mathbf{B}_i^{-1}\mathbf{P}_j - c_j(t) \geq 0, \text{ for all } j$$

The value of t_{i+1} equals the largest $t > t_i$ that satisfies all the optimality conditions.

Note that *nothing* in the inequalities requires **C**(t) to be linear in t. Any function **C**(t), linear or nonlinear, is acceptable. However, with nonlinearity the numerical manipulation of the resulting inequalities may be cumbersome. (See Problem 5, Set 7.5a for an illustration of the nonlinear case.)

Example 7.5-1

$$\text{Maximize } z = (3 - 6t)x_1 + (2 - 2t)x_2 + (5 + 5t)x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 40 \\ 3x_1 \qquad + 2x_3 &\leq 60 \\ x_1 + 4x_2 \qquad &\leq 30 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

We have

$$\mathbf{C}(t) = (3 - 6t, 2 - 2t, 5 + 5t), t \geq 0$$

The variables x_4, x_5, and x_6 will be used as the slack variables associated with the three constraints.

Optimal Solution at $t = t_0 = 0$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	160
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	5
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	30
x_6	2	0	0	-2	1	1	10

$$\mathbf{X}_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$
$$\mathbf{C}_{B_0}(t) = (2 - 2t, 5 + 5t, 0)$$
$$\mathbf{B}_0^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

The optimality conditions for the current nonbasic vectors, $\mathbf{P}_1$, $\mathbf{P}_4$, and $\mathbf{P}_5$, are

$$\{\mathbf{C}_{B_0}(t)\mathbf{B}_0^{-1}\mathbf{P}_j - c_j(t)\}_{j=1,4,5} = (4 + 14t, 1 - t, 2 + 3t) \geq \mathbf{0}$$

Thus, $\mathbf{X}_{B_0}$ remains optimal so long as the following conditions are satisfied:

$$4 + 14t \geq 0$$
$$1 - t \geq 0$$
$$2 + 3t \geq 0$$

Because $t \geq 0$, the second inequality gives $t \leq 1$ and the remaining two inequalities are satisfied for all $t \geq 0$. We thus have $t_1 = 1$, which means that $\mathbf{X}_{B_0}$ remains optimal (and feasible) for $0 \leq t \leq 1$.

The reduced cost $z_4(t) - c_4(t) = 1 - t$ equals zero at $t = 1$ and becomes negative for $t > 1$. Thus, $\mathbf{P}_4$ must enter the basis for $t > 1$. In this case, $\mathbf{P}_2$ must leave the basis (see the optimal tableau at $t = 0$). The new basic solution $\mathbf{X}_{B_1}$ is the alternative solution obtained at $t = 1$ by letting $\mathbf{P}_4$ enter the basis—that is, $\mathbf{X}_{B_1} = (x_4, x_3, x_6)^T$ and $\mathbf{B}_1 = (\mathbf{P}_4, \mathbf{P}_3, \mathbf{P}_6)$.

Alternative Optimal Basis at $t = t_1 = 1$

$$\mathbf{B}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B}_1^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$\mathbf{X}_{B_1} = (x_4, x_3, x_6)^T = \mathbf{B}_1^{-1}\mathbf{b} = (10, 30, 30)^T$$
$$\mathbf{C}_{B_1}(t) = (0, 5 + 5t, 0)$$

The associated nonbasic vectors are $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_5$, and we have

$$\{\mathbf{C}_{B_1}(t)\mathbf{B}_1^{-1}\mathbf{P}_j - c_j(t)\}_{j=1,2,5} = \left(\tfrac{9+27t}{2}, -2+2t, \tfrac{5+5t}{2}\right) \geq \mathbf{0}$$

According to these conditions, the basic solution $\mathbf{X}_{B_1}$ remains optimal for all $t \geq 1$. Observe that the optimality condition, $-2 + 2t \geq 0$, automatically "remembers" that $\mathbf{X}_{B_1}$ is optimal for a range of t that starts from the last critical value $t_1 = 1$. This will always be the case in parametric programming computations.

The optimal solution for the entire range of t is summarized below. The value of z is computed by direct substitution.

t	x_1	x_2	x_3	z
$0 \leq t \leq 1$	0	5	30	$160 + 140t$
$t \geq 1$	0	0	30	$150 + 150t$

PROBLEM SET 7.5A

*1. In example 7.5-1, suppose that t is unrestricted in sign. Determine the range of t for which $\mathbf{X}_{B_0}$ remains optimal.

2. Solve Example 7.5-1, assuming that the objective function is given as
   *(a) Maximize $z = (3 + 3t)x_1 + 2x_2 + (5 - 6t)x_3$
   (b) Maximize $z = (3 - 2t)x_1 + (2 + t)x_2 + (5 + 2t)x_3$
   (c) Maximize $z = (3 + t)x_1 + (2 + 2t)x_2 + (5 - t)x_3$

3. Study the variation in the optimal solution of the following parameterized LP given $t \geq 0$.

$$\text{Minimize } z = (4 - t)x_1 + (1 - 3t)x_2 + (2 - 2t)x_3$$

subject to

$$3x_1 + x_2 + 2x_3 = 3$$
$$4x_1 + 3x_2 + 2x_3 \geq 6$$
$$x_1 + 2x_2 + 5x_3 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

4. The analysis in this section assumes that the optimal solution of the LP at $t = 0$ is obtained by the (primal) simplex method. In some problems, it may be more convenient to obtain the optimal solution by the dual simplex method (Section 4.4.1). Show how the parametric analysis can be carried out in this case, then analyze the LP of Example 4.4-1, assuming that the objective function is given as

$$\text{Minimize } z = (3 + t)x_1 + (2 + 4t)x_2 + x_3, t \geq 0$$

*5. In Example 7.5-1, suppose that the objective function is nonlinear in $t (t \geq 0)$ and is defined as

$$\text{Maximize } z = (3 + 2t^2)x_1 + (2 - 2t^2)x_2 + (5 - t)x_3$$

Determine the first critical value t_1.

7.5.2 Parametric Changes in b

The parameterized right-hand side $\mathbf{b}(t)$ can affect only the feasibility of the problem. The critical values of t are thus determined from the following condition:

$$\mathbf{X}_B(t) = \mathbf{B}^{-1}\mathbf{b}(t) \geq \mathbf{0}$$

Example 7.5-2

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + \ \ x_3 &\leq 40 - t \\ 3x_1 \qquad\quad + 2x_3 &\leq 60 + 2t \\ x_1 + 4x_2 \qquad\quad &\leq 30 - 7t \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Assume that $t \geq 0$.

At $t = t_0 = 0$, the problem is identical to that of Example 7.5-1. We thus have

$$\mathbf{X}_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$

$$\mathbf{B}_0^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

To determine the first critical value t_1, we apply the feasibility conditions $\mathbf{X}_{B_0}(t) = \mathbf{B}_0^{-1}\mathbf{b}(t) \geq 0$, which yields

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 5 - t \\ 30 + t \\ 10 - 3t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

These inequalities are satisfied for $t \leq \frac{10}{3}$, meaning that $t_1 = \frac{10}{3}$ and that the basis $\mathbf{B}_0$ remains feasible for the range $0 \leq t \leq \frac{10}{3}$. However, the values of the basic variables x_2, x_3, and x_6 will change with t as given above.

The value of the basic variable x_6 $(= 10 - 3t)$ will equal zero at $t = t_1 = \frac{10}{3}$, and will become negative for $t > \frac{10}{3}$. Thus, at $t = \frac{10}{3}$, we can determine the alternative basis $\mathbf{B}_1$ by applying the revised dual simplex method (see Problem 5, Set 7.2b, for details). The leaving variable is x_6.

Alternative Basis at $t = t_1 = \frac{10}{3}$

Given that x_6 is the leaving variable, we determine the entering variable as follows:

$$\mathbf{X}_{B_0} = (x_2, x_3, x_6)^T, \mathbf{C}_{B_0} = (2, 5, 0)$$

Thus,

$$\{z_j - c_j\}_{j=1,4,5} = \{\mathbf{C}_{B_0}\mathbf{B}_0^{-1}\mathbf{P}_j - c_j\}_{j=1,4,5} = (4, 1, 2)$$

Next, for nonbasic x_j, $j = 1, 4, 5$, we compute

$$\begin{aligned}(\text{Row of } \mathbf{B}_0^{-1} \text{ associated with } x_6)(\mathbf{P}_1, \mathbf{P}_4, \mathbf{P}_5) &= (\text{Third row of } \mathbf{B}_0^{-1})(\mathbf{P}_1, \mathbf{P}_4, \mathbf{P}_5)\\ &= (-2, 1, 1)(\mathbf{P}_1, \mathbf{P}_4, \mathbf{P}_5)\\ &= (2, -2, 1)\end{aligned}$$

The entering variable is thus associated with

$$\theta = \min\left\{-, \left|\frac{1}{-2}\right|, -\right\} = \frac{1}{2}$$

Thus, $\mathbf{P}_4$ is the entering vector. The alternative basic solution and its $\mathbf{B}_1$ and $\mathbf{B}_1^{-1}$ are

$$\mathbf{X}_{B_1} = (x_2, x_3, x_4)^T$$

$$\mathbf{B}_1 = (\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4) = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix}, \mathbf{B}_1^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

The next critical value t_2 is determined from the feasibility conditions, $\mathbf{X}_{B_1}(t) = \mathbf{B}_1^{-1}\mathbf{b}(t) \geq \mathbf{0}$, which yields

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{30 - 7t}{4} \\ 30 + t \\ \frac{-10 + 3t}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

These conditions show that $\mathbf{B}_1$ remains feasible for $\frac{10}{3} \leq t \leq \frac{30}{7}$.

At $t = t_2 = \frac{30}{7}$, an alternative basis can be obtained by the revised dual simplex method. The leaving variable is x_2, because it corresponds to the condition yielding the critical value t_2.

Alternative Basis at $t = t_2 = \frac{30}{7}$

Given that x_2 is the leaving variable, we determine the entering variable as follows:

$$\mathbf{X}_{B_1} = (x_2, x_3, x_4)^T, \mathbf{C}_{B_1} = (2, 5, 0)$$

Thus,

$$\{z_j - c_j\}_{j=1,5,6} = \{\mathbf{C}_{B_1}\mathbf{B}_1^{-1}\mathbf{P}_j - c_j\}_{j=1,5,6} = \left(5, \tfrac{5}{2}, \tfrac{1}{2}\right)$$

Next, for nonbasic x_j, $j = 1, 5, 6$, we compute

$$\begin{aligned}(\text{Row of } \mathbf{B}_1^{-1} \text{ associated with } x_2)(\mathbf{P}_1, \mathbf{P}_5, \mathbf{P}_6) &= (\text{First row of } \mathbf{B}_1^{-1})(\mathbf{P}_1, \mathbf{P}_5, \mathbf{P}_6)\\ &= \left(0, 0, \tfrac{1}{4}\right)(\mathbf{P}_1, \mathbf{P}_5, \mathbf{P}_6)\\ &= \left(\tfrac{1}{4}, 0, \tfrac{1}{4}\right)\end{aligned}$$

Because all the denominator elements, $\left(\frac{1}{4}, 0, \frac{1}{4}\right)$, are ≥ 0, the problem has no feasible solution for $t > \frac{30}{7}$ and the parametric analysis ends at $t = t_2 = \frac{30}{7}$.

The optimal solution is summarized as

t	x_1	x_2	x_3	z
$0 \le t \le \frac{10}{3}$	0	$5 - t$	$30 + t$	$160 + 3t$
$\frac{10}{3} \le t \le \frac{30}{7}$	0	$\frac{30 - 7t}{4}$	$30 + t$	$165 + \frac{3}{2}t$
$t > \frac{30}{7}$	(No feasible solution exists)			

PROBLEM SET 7.5B

*1. In Example 7.5-2, find the first critical value, t_1, and define the vectors of $\mathbf{B}_1$ in each of the following cases:

*(a) $\mathbf{b}(t) = (40 + 2t, 60 - 3t, 30 + 6t)^T$

(b) $\mathbf{b}(t) = (40 - t, 60 + 2t, 30 - 5t)^T$

*2. Study the variation in the optimal solution of the following parameterized LP, given $t \ge 0$.

$$\text{Minimize } z = 4x_1 + x_2 + 2x_3$$

subject to

$$3x_1 + x_2 + 2x_3 = 3 + 3t$$
$$4x_1 + 3x_2 + 2x_3 \ge 6 + 2t$$
$$x_1 + 2x_2 + 5x_3 \le 4 - t$$
$$x_1, x_2, x_3 \ge 0$$

3. The analysis in this section assumes that the optimal LP solution at $t = 0$ is obtained by the (primal) simplex method. In some problems, it may be more convenient to obtain the optimal solution by the dual simplex method (Section 4.4.1). Show how the parametric analysis can be carried out in this case, and then analyze the LP of Example 4.4-1, assuming that $t \ge 0$ and the right-hand side vector is

$$\mathbf{b}(t) = (3 + 2t, 6 - t, 3 - 4t)^T$$

4. Solve Problem 2 assuming that the right-hand side is changed to

$$\mathbf{b}(t) = (3 + 3t^2, 6 + 2t^2, 4 - t^2)^T$$

Further assume that t can be positive, zero, or negative.

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