CHAPTER 14

Probabilistic Inventory Models

Chapter Guide. This chapter is a continuation of the material in Chapter 11 on deterministic inventory models. It deals with inventory situations in which the demand is probabilistic. The developed models are categorized broadly under *continuous* and *periodic* review situations. The periodic review models include both single-period and multiperiod cases. The proposed solutions range from the use of a probabilistic version of the deterministic EOQ to more complex situations solved by dynamic programming. It may appear that the probabilistic models presented here are "too theoretical" to be practical. But, in fact, a case analysis in Chapter 24 on the CD uses one of these models to help Dell, Inc. manage its inventory situation and realize sizable savings.

This chapter includes a summary of 1 real-life application, 4 solved examples, 1 Excel template, 22 end-of-section problems, and 2 cases. The cases are in Appendix E on the CD. The AMPL/Excel/Solver/TORA programs are in folder ch14Files.

Real-Life Application—Inventory Decisions in Dell's Supply Chain

Dell, Inc., implements a direct-sales business model in which personal computers are sold directly to customers in the United States. When an order arrives from a customer, the specifications are sent to a manufacturing plant in Austin, Texas, where the computer is built, tested, and packaged in about eight hours. Dell carries little inventory. Instead, its suppliers, normally located in Southeast Asia, are required to keep what is known as "revolving" inventory on hand in *revolvers* (warehouses) near the manufacturing plants. These revolvers are owned by Dell and leased to the suppliers. Dell then "pulls" parts as needed from the revolvers, and it is the suppliers' responsibility to replenish the inventory to meet Dell's forecasted demand. Although Dell does not own the inventory in the revolvers, its cost is indirectly passed on to customers through component pricing. Thus, any reduction in inventory directly benefits Dell's customers by reducing product prices. The proposed solution has resulted in an estimated \$2.7 million in annual savings. Case 13 in Chapter 24 on the CD provides the details of the study.

14.1 CONTINUOUS REVIEW MODELS

This section presents two models: (1) a "probabilitized" version of the deterministic EOQ (Section 11.2.1) that uses a buffer stock to account for probabilistic demand, and (2) a more exact probabilistic EOQ model that includes the probabilistic demand directly in the formulation.

14.1.1 "Probabilitized" EOQ Model

Some practitioners have sought to adapt the deterministic EOQ model (Section 11.2.1) to reflect the probabilistic nature of demand by using an approximation that superimposes a constant buffer stock on the inventory level throughout the entire planning horizon. The size of the buffer is determined such that the probability of running out of stock *during lead time* (the period between placing and receiving an order) does not exceed a prespecified value.

Let

L = Lead time between placing and receiving an order

x_L = Random variable representing demand during lead time

μ_L = Average demand during lead time

σ_L = Standard deviation of demand during lead time

B = Buffer stock size

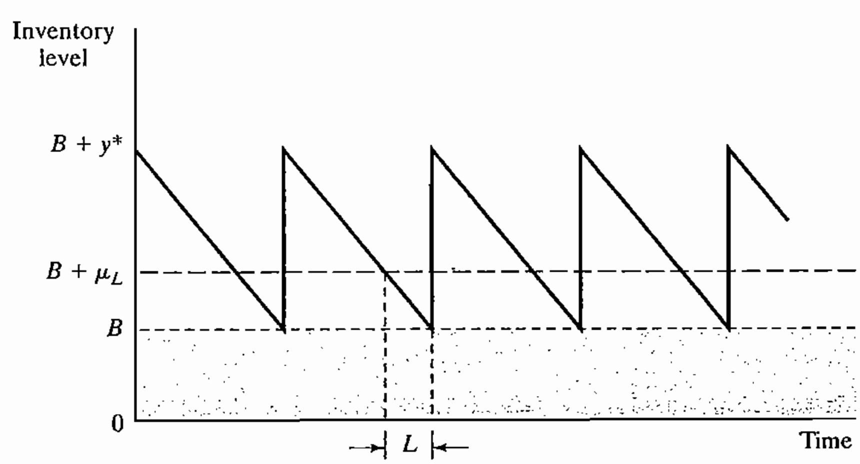
α = Maximum allowable probability of running out of stock during lead time

The main assumption of the model is that the demand, x_L, during lead time L is normally distributed with mean μ_L and standard deviation σ_L—that is, $N(\mu_L, \sigma_L)$.

Figure 14.1 depicts the relationship between the buffer stock, B, and the parameters of the deterministic EOQ model that include the lead time L, the average demand

FIGURE 14.1

Buffer stock imposed on the classical EOQ model



during lead time, μ_L, and the EOQ, y^*. Note that L must equal the *effective* lead time as defined in Section 11.2.1.

The probability statement used to determine B can be written as

$$P\{x_L \geq B + \mu_L\} \leq \alpha$$

We can convert x_L into a standard $N(0, 1)$ random variable by using the following substitution (see Section 12.5.4):

$$z = \frac{x_L - \mu_L}{\sigma_L}$$

Thus, we have

$$P\left\{z \geq \frac{B}{\sigma_L}\right\} \leq \alpha$$

Figure 14.2 defines K_α (which is determined from the standard normal tables in Appendix B or by using file excelStatTables.xls) such that

$$P\{z \geq K_\alpha\} = \alpha$$

Hence, the buffer size must satisfy

$$B \geq \sigma_L K_\alpha$$

The demand during the lead time L usually is described by a probability density function *per unit time* (e.g., per day or week), from which the distribution of the demand during L can be determined. Given that the demand per unit time is normal with mean D and standard deviation σ, the mean and standard deviation, μ_L and σ_L, of demand during lead time, L, are computed as

$$\mu_L = DL$$
$$\sigma_L = \sqrt{\sigma^2 L}$$

The formula for σ_L requires L to be (rounded to) an integer value.

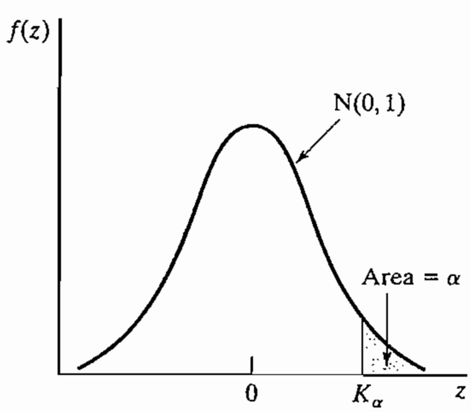


FIGURE 14.2

Probability of running out of stock, $P\{z \geq K_\alpha\} = \alpha$

Example 14.1-1

In Example 11.2-1 dealing with determining the inventory policy of neon lights, EOQ = 1000 *units*. If the *daily* demand is normal with mean $D = 100$ lights and standard deviation $\sigma = 10$ lights—that is, $N(100, 10)$—determine the buffer size so that the probability of running out of stock is below $\alpha = .05$.

From Example 11.2-1, the effective lead time is $L = 2$ days. Thus,

$$\mu_L = DL = 100 \times 2 = 200 \text{ units}$$

$$\sigma_L = \sqrt{\sigma^2 L} = \sqrt{10^2 \times 2} = 14.14 \text{ units}$$

Given $K_{.05} = 1.645$, the buffer size is computed as

$$B \geq 14.14 \times 1.645 \approx 23 \text{ neon lights}$$

Thus, the optimal inventory policy with buffer B calls for ordering 1000 units whenever the inventory level drops to 223 ($= B + \mu_L = 23 + 2 \times 100$) units.

PROBLEM SET 14.1A

1. In Example 14.1-1, determine the optimal inventory policy for each of the following cases:
   *(a) Lead time = 15 days.
   (b) Lead time = 23 days.
   (c) Lead time = 8 days.
   (d) Lead time = 10 days.
2. A music store sells a best-selling compact disc. The daily demand (in number of units) for the disc is approximately normally distributed with mean 200 discs and standard deviation 20 discs. The cost of keeping the discs in the store is \$.04 per disc per day. It costs the store \$100 to place a new order. There is a 7-day lead time for delivery. Assuming that the store wants to limit the probability of running out of discs during the lead time to no more than .02, determine the store's optimal inventory policy.
3. The daily demand for camera films at a gift shop in a resort area is normally distributed with mean 300 rolls and standard deviation 5 rolls. The cost of holding a roll in the shop is \$.02. A fixed cost of \$30 is incurred each time a new order of films is placed by the shop. The shop's inventory policy calls for ordering 150 rolls whenever the inventory level drops to 80 units while simultaneously maintaining a constant buffer of 20 rolls at all times.
   (a) For the stated inventory policy, determine the probability of running out of stock during lead time.
   (b) Given the data of the situation, recommend an inventory policy for the shop, assuming that the probability of running out of films during the lead time does not exceed .10.

14.1.2 Probabilistic EOQ Model

There is no reason to believe that the "probabilitized" EOQ model in Section 14.1.1 will produce an optimal inventory policy. The fact that pertinent information regarding the probabilistic nature of demand is initially ignored, only to be "revived" in a totally independent manner at a later stage of the calculations, is sufficient to refute optimality.

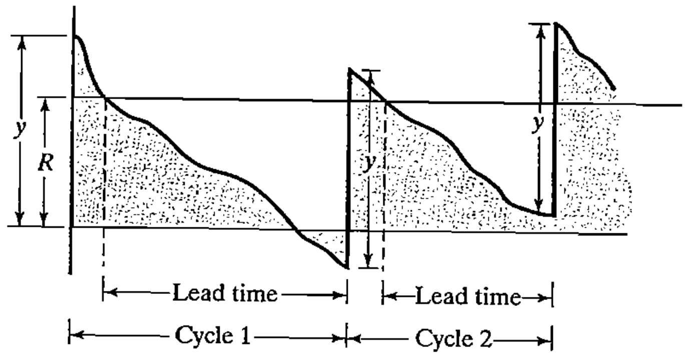


FIGURE 14.3
Probabilistic inventory model with shortage

To remedy the situation, a more accurate model is presented in which the probabilistic nature of the demand is included directly in the formulation of the model.

Unlike the case in Section 14.1.1, the new model allows shortage of demand, as Figure 14.3 demonstrates. The policy calls for ordering the quantity y whenever the inventory drops to level R. As in the deterministic case, the reorder level R is a function of the lead time between placing and receiving an order. The optimal values of y and R are determined by minimizing the expected cost per unit time that includes the sum of the setup, holding, and shortage costs.

The model has three assumptions.

1. Unfilled demand during lead time is backlogged.
2. No more than one outstanding order is allowed.
3. The distribution of demand during lead time remains stationary (unchanged) with time.

To develop the total cost function per unit time, let

$$f(x) = \text{pdf of demand, } x\text{, during lead time}$$
$$D = \text{Expected demand per unit time}$$
$$h = \text{Holding cost per inventory unit per unit time}$$
$$p = \text{Shortage cost per inventory unit}$$
$$K = \text{Setup cost per order}$$

Based on these definitions, the elements of the cost function are now determined.

1. *Setup cost.* The approximate number of orders per unit time is $\frac{D}{y}$, so that the setup cost per unit time is approximately $\frac{KD}{y}$.

2. *Expected holding cost.* The average inventory is

$$I = \frac{(y + E\{R - x\}) + E\{R - x\}}{2} = \frac{y}{2} + R - E\{x\}$$

The formula is based on the average of the beginning and ending expected inventories of a cycle, $y + E\{R - x\}$ and $E\{R - x\}$, respectively. As an approximation, the expression ignores the case where $R - E\{x\}$ may be negative. The expected holding cost per unit time thus equals hI.

3. *Expected shortage cost.* Shortage occurs when $x > R$. Thus, the expected shortage quantity per cycle is

$$S = \int_R^\infty (x - R)f(x)\,dx$$

Because p is assumed to be proportional to the shortage quantity only, the expected shortage cost per cycle is pS, and, based on $\frac{D}{y}$ cycles per unit time, the shortage cost per unit time is $\frac{pDS}{y}$.

The resulting total cost function per unit time is

$$\text{TCU}(y, R) = \frac{DK}{y} + h\left(\frac{y}{2} + R - E\{x\}\right) + \frac{pD}{y}\int_R^\infty (x - R)f(x)\,dx$$

The solutions for optimal y^* and R^* are determined from

$$\frac{\partial \text{TCU}}{\partial y} = -\left(\frac{DK}{y^2}\right) + \frac{h}{2} - \frac{pDS}{y^2} = 0$$

$$\frac{\partial \text{TCU}}{\partial R} = h - \left(\frac{pD}{y}\right)\int_R^\infty f(x)\,dx = 0$$

We thus get

$$y^* = \sqrt{\frac{2D(K + pS)}{h}} \tag{1}$$

$$\int_{R^*}^\infty f(x)\,dx = \frac{hy^*}{pD} \tag{2}$$

Because y^* and R^* cannot be determined in closed forms from (1) and (2), a numeric algorithm, developed by Hadley and Whitin (1963, pp. 169–174), is used to find the solutions. The algorithm converges in a finite number of iterations, provided a feasible solution exists.

For $R = 0$, (1) and (2) above yield

$$\hat{y} = \sqrt{\frac{2D(K + pE\{x\})}{h}}$$

$$\tilde{y} = \frac{PD}{h}$$

If $\tilde{y} \geq \hat{y}$, unique optimal values of y and R exist. The solution procedure recognizes that the smallest value of y^* is $\sqrt{\frac{2KD}{h}}$, which is achieved when $S = 0$.

The steps of the algorithm are

Step 0. Use the initial solution $y_1 = y^* = \sqrt{\frac{2KD}{h}}$, and let $R_0 = 0$. Set $i = 1$, and go to step i.

Step i. Use y_i to determine R_i from Equation (2). If $R_i \approx R_{i-1}$, stop; the optimal solution is $y^* = y_i$, and $R^* = R_i$. Otherwise, use R_i in Equation (1) to compute y_i. Set $i = i + 1$, and repeat step i.

Example 14.1-2

Electro uses resin in its manufacturing process at the rate of 1000 gallons per month. It costs Electro \$100 to place an order for a new shipment. The holding cost per gallon per month is \$2, and the shortage cost per gallon is \$10. Historical data show that the demand during lead time is uniform over the range (0, 100) gallons. Determine the optimal ordering policy for Electro.

Using the symbols of the model, we have

$$\begin{aligned} D &= 1000 \text{ gallons per month} \\ K &= \$100 \text{ per order} \\ h &= \$2 \text{ per gallon per month} \\ p &= \$10 \text{ per gallon} \\ f(x) &= \tfrac{1}{100},\ 0 \le x \le 100 \\ E\{x\} &= 50 \text{ gallons} \end{aligned}$$

First, we need to check whether the problem has a feasible solution. Using the equations for $\hat{y}$ and $\tilde{y}$ we get

$$\hat{y} = \sqrt{\frac{2 \times 1000(100 + 10 \times 50)}{2}} = 774.6 \text{ gallons}$$

$$\tilde{y} = \frac{10 \times 1000}{2} = 5000 \text{ gallons}$$

Because $\tilde{y} \ge \hat{y}$, a unique solution exists for y^* and R^*.

The expression for S is computed as

$$S = \int_R^{100} (x - R)\frac{1}{100}dx = \frac{R^2}{200} - R + 50$$

Using S in Equations (1) and (2), we obtain

$$y_i = \sqrt{\frac{2 \times 1000(100 + 10S)}{2}} = \sqrt{100{,}000 + 10{,}000S} \text{ gallons} \qquad (3)$$

$$\int_R^{100} \frac{1}{100}dx = \frac{2y_i}{10 \times 1000}$$

The last equation yields

$$R_i = 100 - \frac{y_i}{50} \quad (4)$$

We now use Equations (3) and (4) to determine the solution.

Iteration 1

$$y_1 = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \times 1000 \times 100}{2}} = 316.23 \text{ gallons}$$

$$R_1 = 100 - \frac{316.23}{50} = 93.68 \text{ gallons}$$

Iteration 2

$$S = \frac{R_1^2}{200} - R_1 + 50 = .19971 \text{ gallons}$$

$$y_2 = \sqrt{100{,}000 + 10{,}000 \times .19971} = 319.37 \text{ gallons}$$

Hence,

$$R_2 = 100 - \frac{319.39}{50} = 93.612$$

Iteration 3

$$S = \frac{R_2^2}{200} - R_2 + 50 = .20399 \text{ gallon}$$

$$y_3 = \sqrt{100{,}000 + 10{,}000 \times .20399} = 319.44 \text{ gallons}$$

Thus,

$$R_3 = 100 - \frac{319.44}{50} = 93.611 \text{ gallons}$$

Because $y_3 \approx y_2$ and $R_3 \approx R_2$, the optimum is $R^* \approx 93.611$ gallons, $y^* \approx 319.44$ gallons. File excelContRev.xls can be used to determine the solution to any desired degree of accuracy. The optimal inventory policy calls for ordering approximately 320 gallons whenever the inventory level drops to 94 gallons.

PROBLEM SET 14.1B

1. For the data given in Example 14.1-2, determine the following:
   (a) The approximate number of orders per month.
   (b) The expected monthly setup cost.
   (c) The expected holding cost per month.
   (d) The expected shortage cost per month.
   (e) The probability of running out of stock during lead time.

*2. Solve Example 14.1-2, assuming that the demand during lead time is uniform between 0 and 50 gallons.

***3.** In Example 14.1-2, suppose that the demand during lead time is uniform between 40 and 60 gallons. Compare the solution with that obtained in Example 14.1-2, and interpret the results. (*Hint:* In both problems $E\{x\}$ is the same, but the variance in the present problem is smaller.)

4. Find the optimal solution for Example 14.1-2, assuming that the demand during lead time is $N(100, 2)$. Assume that $D = 10{,}000$ gallons per month, $h = \$2$ per gallon per month, $p = \$4$ per gallon, and $K = \$20$.

14.2 SINGLE-PERIOD MODELS

Single-item inventory models occur when an item is ordered only once to satisfy the demand for the period. For example, fashion items become obsolete at the end of the season. This section presents two models representing the no-setup and the setup cases.

The symbols used in the development of the models include

K = Setup cost per order

h = Holding cost per held unit during the period

p = Penalty cost per shortage unit during the period

D = Random variable representing demand during the period

$f(D)$ = pdf of demand during the period

y = Order quantity

x = Inventory on hand before an order is placed.

The model determines the optimal value of y that minimizes the sum of the expected holding and shortage costs. Given optimal y $(= y^*)$, the inventory policy calls for ordering $y^* - x$ if $x < y$; otherwise, no order is placed.

14.2.1 No-Setup Model (Newsvendor Model)

This model has come to be known in the literature as the *newsvendor* model (the original classical name is the *newsboy* model) because it deals with items with short life such as newspapers.

The assumptions of this model are

1. Demand occurs instantaneously at the start of the period immediately after the order is received.
2. No setup cost is incurred.

Figure 14.4 demonstrates the inventory position after the demand, D, is satisfied. If $D < y$, the quantity $y - D$ is held during the period. Otherwise, a shortage amount $D - y$ will result if $D > y$.

The expected cost for the period, $E\{C(y)\}$, is expressed as

$$E\{C(y)\} = h\int_0^y (y - D)f(D)\,dD + p\int_y^\infty (D - y)f(D)\,dD$$

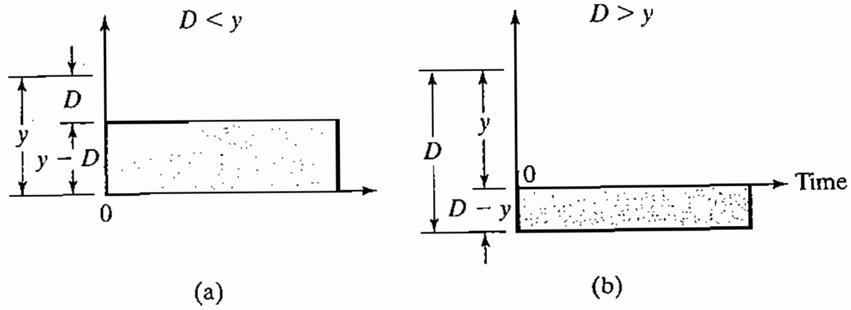


FIGURE 14.4
Holding and shortage inventory in a single-period model

The function $E\{C(y)\}$ can be shown to have a unique minimum because it is convex in y. Taking the first derivative of $E\{C(y)\}$ with respect to y and equating it to zero, we get

$$h\int_0^y f(D)\,dD - p\int_y^\infty f(D)\,dD = 0$$

or

$$hP\{D \le y\} - p(1 - P\{D \le y\}) = 0$$

or

$$P\{D \le y^*\} = \frac{p}{p + h}$$

The preceding development assumes that the demand D is continuous. If D is discrete, then $f(D)$ is defined only at discrete points and the associated cost function is

$$E\{C(y)\} = h\sum_{D=0}^{y}(y - D)f(D) + p\sum_{D=y+1}^{\infty}(D - y)f(D)$$

The necessary conditions for optimality are

$$E\{C(y - 1)\} \ge E\{C(y)\} \text{ and } E\{C(y + 1)\} \ge E\{C(y)\}$$

These conditions are sufficient because $E\{C(y)\}$ is a convex function. After some algebraic manipulations, the application of these conditions yields the following inequalities for determining y^*:

$$P\{D \le y^* - 1\} \le \frac{p}{p + h} \le P\{D \le y^*\}$$

Example 14.2-1

The owner of a newsstand wants to determine the number of *USA Now* newspapers that must be stocked at the start of each day. The owner pays 30 cents for a copy and sells it for 75 cents. The sale of the newspaper typically occurs between 7:00 and 8:00 A.M. Newspapers left at the end of the day are recycled for an income of 5 cents a copy. How many copies should the owner stock every morning, assuming that the demand for the day can be described as

(a) A normal distribution with mean 300 copies and standard deviation 20 copies.

(b) A discrete pdf, $f(D)$, defined as

D	200	220	300	320	340
$f(D)$	.1	.2	.4	.2	.1

The holding and penalty costs are not defined directly in this situation. The data of the problem indicate that each unsold copy will cost the owner $30 - 5 = 25$ cents and that the penalty for running out of stock is $75 - 30 = 45$ cents per copy. Thus, in terms of the parameters of the inventory problem, we have $h = 25$ cents per copy per day and $p = 45$ cents per copy per day.

First, we determine the critical ratio as

$$\frac{p}{p+h} = \frac{45}{45+25} = .643$$

Case (a). The demand D is $N(300, 20)$. We can use excelStatTables.xls to determine the optimum order quantity by entering 300 in F15, 20 in G15, and .643 in L15, which gives the desired answer of 307.33 newspapers in R15. Alternatively, we can use the standard normal tables in Appendix B. Define

$$z = \frac{D - 300}{20}$$

Then from the tables

$$P\{z \leq .366\} \approx .643$$

or

$$\frac{y^* - 300}{20} = .366$$

Thus, $y^* = 307.3$. The optimal order is approximately 308 copies.

Case (b). The demand D follows a discrete pdf, $f(D)$. First, we determine the CDF $P\{D \leq y\}$ as

y	200	220	300	320	340
$P\{D \leq y\}$	.1	.3	.7	.9	1.0

For the computed critical ratio of .643, we have

$$P(D \leq 220) \leq .643 \leq P(D \leq 300)$$

It only follows that $y^* = 300$ copies.

PROBLEM SET 14.2A

1. For the single-period model, show that for the discrete demand the optimal order quantity is determined from

$$P\{D \leq y^* - 1\} \leq \frac{p}{p + h} \leq P\{D \leq y^*\}$$

2. The demand for an item during a single period occurs instantaneously at the start of the period. The associated pdf is uniform between 10 and 15 units. Because of the difficulty in estimating the cost parameters, the order quantity is determined such that the probability of either surplus or shortage does not exceed .1. Is it possible to satisfy both conditions simultaneously?

*3. The unit holding cost in a single-period inventory situation is $1. If the order quantity is 4 units, find the permissible range of the unit penalty cost implied by the optimal conditions. Assume that the demand occurs instantaneously at the start of the period and that demand pdf is given in the following table:

D	0	1	2	3	4	5	6	7	8
$f(D)$	.05	.1	.1	.2	.25	.15	.05	.05	.05

4. The U of A Bookstore offers a program of reproducing class notes for participating professors. Professor Yataha teaches a freshmen-level class, where an enrollment of between 200 and 250 students, uniformly distributed, is expected. It costs the bookstore $10 to produce each copy, which it then sells to the students for $25 a copy. The students purchase their books at the start of the semester. Any unsold copies of Professor Yataha's notes are shredded for recycling. In the meantime, once the bookstore runs out of copies, no additional copies are printed, and the students are responsible for securing the notes from other sources. If the bookstore wants to maximize its revenues, how many copies should it print?

5. QuickStop provides its customers with coffee and donuts at 6:00 A.M. each day. The convenience store buys the donuts for 7 cents apiece and sells them for 25 cents apiece until 8:00 A.M. After 8:00 A.M., the donuts sell for 5 cents apiece. The number of customers buying donuts between 6:00 and 8:00 is uniformly distributed between 30 and 50. Each customer usually orders 3 donuts with coffee. Approximately how many dozen donuts should QuickStop stock every morning to maximize revenues?

*6. Colony Shop is stocking heavy coats for next winter. Colony pays $50 for a coat and sells it for $110. At the end of the winter season, Colony offers the coats at $55 each. The demand for coats during the winter season is more than 20 but less than or equal to 30, all with equal probabilities. Because the winter season is short, the unit holding cost is negligible. Also, Colony's manager does not believe that any penalty would result from coat shortages. Determine the optimal order quantity that will maximize the revenue for Colony Shop. You may use continuous approximation.

7. For the single-period model, suppose that the item is consumed uniformly during the period (rather than instantaneously at the start of the period). Develop the associated cost model, and find the optimal order quantity.

8. Solve Example 14.2-1, assuming that the demand is continuous and uniform during the period and that the pdf of demand is uniform between 0 and 100. (*Hint*: Use the results of Problem 7.)

14.2.2 Setup Model (s-S Policy)

The present model differs from the one in Section 14.2.1 in that a setup cost K is incurred. Using the same notation, the total expected cost per period is

$$E\{\overline{C}(y)\} = K + E\{C(y)\}$$
$$= K + h\int_0^y (y - D)f(D)\,dD + p\int_y^\infty (D - y)f(D)\,dD$$

As shown in Section 14.2.1, the optimum value y^* must satisfy

$$P\{y \le y^*\} = \frac{p}{p + h}$$

Because K is constant, the minimum value of $E\{\overline{C}(y)\}$ must also occur at y^*.

In Figure 14.5, $S = y^*$ and the value of s ($< S$) is determined from the equation

$$E\{C(s)\} = E\{\overline{C}(S)\} = K + E\{C(S)\},\ s < S$$

The equation yields another value s_1 ($> S$), which is discarded.

Given that the amount on hand before an order is placed is x units, how much should be ordered? This question is investigated under three conditions:

1. $x < s$.
2. $s \le x \le S$.
3. $x > S$.

Case 1 ($x < s$). Because x is already on hand, its equivalent cost is given by $E\{C(x)\}$. If any additional amount $y - x$ ($y > x$) is ordered, the corresponding cost given y is $E\{\overline{C}(y)\}$, which includes the setup cost K. From Figure 14.5, we have

$$\min_{y>x} E\{\overline{C}(y)\} = E(\overline{C}(S)) < E\{C(x)\}$$

Thus, the optimal inventory policy in this case is to order $S - x$ units.

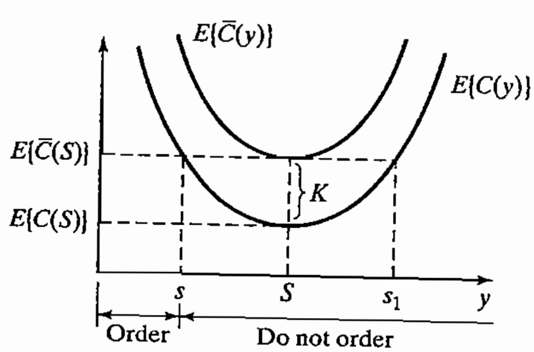


FIGURE 14.5

(s-S) optimal ordering policy in a single-period model with setup cost

Case 2 ($s \le x \le S$). From Figure 14.5, we have

$$E\{C(x)\} \le \min_{y>x} E\{\overline{C}(y)\} = E(\overline{C}(S))$$

Thus, it is *not* advantageous to order in this case. Hence, $y^* = x$.

Case 3 ($x > S$). From Figure 14.5, we have for $y > x$,

$$E\{C(x)\} < E\{\overline{C}(y)\}$$

This condition indicates that it is not advantageous to order in this case—that is, $y^* = x$.

The optimal inventory policy, frequently referred to as the s-S policy, is summarized as

$$\text{If } x < s, \text{ order } S - x$$

$$\text{If } x \ge s, \text{ do not order}$$

The optimality of the s-S policy is guaranteed because the associated cost function is convex.

Example 14.2-2

The daily demand for an item during a single period occurs instantaneously at the start of the period. The pdf of the demand is uniform between 0 and 10 units. The unit holding cost of the item during the period is \$.50, and the unit penalty cost for running out of stock is \$4.50. A fixed cost of \$25 is incurred each time an order is placed. Determine the optimal inventory policy for the item.

To determine y^*, consider

$$\frac{p}{p+h} = \frac{4.5}{4.5 + .5} = .9$$

Also,

$$P\{D \le y^*\} = \int_0^{y^*} \frac{1}{10} dD = \frac{y^*}{10}$$

Thus, $S = y^* = 9$.

The expected cost function is given as

$$E\{C(y)\} = .5\int_0^y \frac{1}{10}(y - D)\,dD + 4.5\int_y^{10} \frac{1}{10}(D - y)\,dD$$

$$= .25y^2 - 4.5y + 22.5$$

The value of s is determined by solving

$$E\{C(s)\} = K + E\{C(S)\}$$

This yields

$$.25s^2 - 4.5s + 22.5 = 25 + .25S^2 - 4.5S + 22.5$$

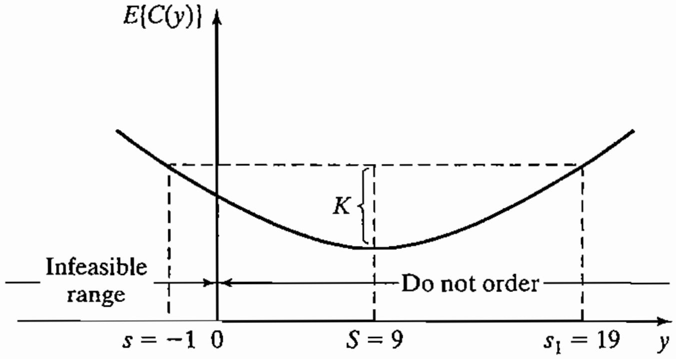


FIGURE 14.6
s-S policy applied to Example 14.2-2

Given $S = 9$, the preceding equation reduces to

$$s^2 - 18s - 19 = 0$$

The solution of this equation is $s = -1$ or $s = 19$. The value of $s > S$ is discarded. Because the remaining value is negative $(= -1)$, s has no feasible value (Figure 14.6). This conclusion usually happens when the cost function is "flat" or when the setup cost is high relative to the other costs of the model.

PROBLEM SET 14.2B

*1. Determine the optimal inventory policy for the situation in Example 14.2-2, assuming that the setup cost is $5.

2. In the single-period model in Section 14.2.1, suppose instead that the model maximizes profit and that a setup cost K is incurred. Given that r is the unit selling price and using the information in Section 14.2.1, develop an expression for the expected profit and determine the optimal order quantity. Solve the problem numerically for $r = \$3$, $c = \$2$, $p = \$4$, $h = \$1$, and $K = \$10$. The demand pdf is uniform between 0 and 10.

3. Work Problem 5, Set 14.2a, assuming that there is a fixed cost of $10 associated with the delivery of donuts.

14.3 MULTIPERIOD MODEL

This section presents a multiperiod model under the assumption of no setup cost. Additionally, the model allows backlog of demand and assumes a zero-delivery lag. It further assumes that the demand D in any period is described by a stationary pdf $f(D)$.

The multiperiod model considers the discounted value of money. If α (< 1) is the discount factor per period, then an amount $\$A$ available n periods from now has a present value of $\$\alpha^n A$.

Suppose that the inventory situation encompasses n periods and that unfilled demand can be backlogged exactly one period. Define

$$F_i(x_i) = \text{Maximum expected profit for periods } i, i+1, \ldots, \text{ and } n, \text{ given that } x_i \text{ is the amount on hand before an order is placed in period } i$$

Using the notation in Section 14.2 and assuming that c and r are the cost and revenue per unit, respectively, the inventory situation can be formulated using the following dynamic programming model (see Chapter 22 on the CD):

$$F_i(x_i) = \max_{y_i \geq x_i}\left\{-c(y_i - x_i) + \int_0^{y_i}[rD - h(y_i - D)]f(D)\,dD \right.$$
$$+ \int_{y_i}^{\infty}[ry_i + \alpha r(D - y_i) - p(D - y_i)]f(D)\,dD$$
$$\left. + \alpha\int_0^{\infty}F_{i+1}(y_i - D)f(D)\,dD\right\}, i = 1, 2, \ldots, n$$

where $F_{n+1}(y_n - D) = 0$. The value of x_i may be negative because unfilled demand is backlogged. The quantity $\alpha r(D - y_i)$ in the second integral is included because $(D - y_i)$ is the unfilled demand in period i that must be filled in period $i + 1$.

The problem can be solved recursively. For the case where the number of periods is infinite, the recursive equation reduces to

$$F(x) = \max_{y \geq x}\left\{-c(y - x) + \int_0^{y}[rD - h(y - D)]f(D)\,dD \right.$$
$$+ \int_{y}^{\infty}[ry + \alpha r(D - y) - p(D - y)]f(D)\,dD$$
$$\left. + \alpha\int_0^{\infty}F(y - D)f(D)\,dD\right\}$$

where x and y are the inventory levels for each period before and after an order is received, respectively.

The optimal value of y can be determined from the following necessary condition, which also happens to be sufficient because the expected revenue function $F(x)$ is concave.

$$\frac{\partial(.)}{\partial y} = -c - h\int_0^{y}f(D)\,dD + \int_y^{\infty}[(1 - \alpha)r + p]f(D)\,dD$$
$$+ \alpha\int_0^{\infty}\frac{\partial F(y - D)}{\partial y}f(D)\,dD = 0$$

The value of $\frac{\partial F(y - D)}{\partial y}$ is determined as follows. If there are β (> 0) more units on hand at the start of the next period, the profit for the next period will increase by $c\beta$, because this much less has to be ordered. This means that

$$\frac{\partial F(y - D)}{\partial y} = c$$

The necessary condition thus becomes

$$-c - h\int_0^y f(D)\,dD + [(1-\alpha)r + p]\left(1 - \int_0^y f(D)\,dD\right) + \alpha c\int_0^\infty f(D)\,dD = 0$$

The optimum inventory level y^* is thus determined from

$$\int_0^{y^*} f(D)\,dD = \frac{p + (1-\alpha)(r-c)}{p + h + (1-\alpha)r}$$

The optimal inventory policy for each period, given its entering inventory level x, is thus given as

If $x < y^*$, order $y^* - x$
If $x \geq y^*$, do not order

PROBLEM SET 14.3A

1. Consider a two-period probabilistic inventory model in which the demand is backlogged, and orders are received with zero delivery lag. The demand pdf per period is uniform between 0 and 10, and the cost parameters are given as

Unit selling price = \$2

Unit purchase price = \$1

Unit holding cost per month = \$.10

Unit penalty cost per month = \$3

Discount factor = .8

Find the optimal inventory policy for the two periods, assuming that the initial inventory for period 1 is zero.

***2.** The pdf of the demand per period in an infinite-horizon inventory model is given as

$$f(D) = .08D, 0 \leq D \leq 5$$

The unit cost parameters are

Unit selling price = \$10

Unit purchase price = \$8

Unit holding cost per month = \$1

Unit penalty cost per month = \$10

Discount factor = .9

Determine the optimal inventory policy assuming zero delivery lag and that the unfilled demand is backlogged.

3. Consider the infinite-horizon inventory situation with zero delivery lag and backlogged demand. Develop the optimal inventory policy based on the minimization of cost given that

$$\text{Holding cost for } z \text{ units} = hz^2$$

$$\text{Penalty cost for } z \text{ units} = px^2$$

Show that for the special case where $h = p$, the optimal solution is independent of pdf of demand.

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